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THE UNIVERSITY OF ALBERTA

AN ANALYSIS OF MICROCOMPUTER USE IN GRADE ELEVEN MATHEMATICS

by

©

Karen L. Miklos

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF EDUCATION

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA
SPRING 1986

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Thomas E. Kieren
Supervisor

Joan Worth

Elton T. Olson

Date 10 January, 1986

Abstract

The search for effective ways to integrate microcomputers into classroom instruction is of particular interest to mathematics teachers. Although mathematics and computers bear a close relationship, the use of the technology in teaching mathematical concepts is as challenging in this subject area as in any other. The purpose of this study was three-fold: (1) to adapt a concept of instruction for understanding in mathematics to include the notion that computer algorithms may be used to represent mathematical relationships, (2) to develop a set of materials for using computer programming in teaching a unit on quadratic functions, and (3) to observe and analyze student productions in an actual classroom setting. The basic research question focused on whether the integration of computer programming with regular classroom instruction would enhance students' understanding of mathematics concepts in terms of the completeness and "correctness" of their representation and their ability to relate correctly new and existing knowledge.

In order to address the research question, a unit on quadratic functions was developed in which computer programming was applied in teaching concepts such as graphing functions, examining roots, and using the discriminant. Computers were used in the study to present

graphic images of quadratic functions and to construct algorithms to represent quadratic concepts and computations. Provisions were made in the unit for the use of both commercial courseware and student programs written in BASIC. The materials were tested with a group of honors Mathematics 20 students. The study was conducted in parallel classroom sessions and computer laboratory sessions. The teacher provided the regular classroom instruction, and the researcher served as the instructor for the laboratory sessions in which Apple IIe computers were used. The study required about 1450 minutes of class time, including the time required to teach a computer literacy unit. Approximately half the time (700 minutes) was spent in the computer laboratory.

Analysis of the results of the field test were based on notes recorded daily by the researcher, a pre-test and a post-test of concepts related to quadratic functions, a student questionnaire, and interviews with the teacher. The results tended to confirm that computer programming helped students to increase their understanding of basic concepts and to overcome some misconceptions. Through the writing of computer programs, students realized the need to specify the order of operation when working with formulas. Students' understanding of how maximum and minimum values are determined was also clarified. In addition, students began to see relationships among concepts as well as connectedness between graphing and solving problems. The process of writing and correcting computer programs actively involved the students in the learning process; indications were that such involvement enhanced understanding.

Student and teacher responses reflected favorable attitudes toward computer programming, although consensus was absent on some issues. Among the unresolved questions was the specific approach which should be taken to integrating computers with regular instruction. The procedure of parallel computer use and regular instruction in the teaching of a mathematical concept, as was used in this study, was questioned by both the teacher and the students. The approach taken in this study confirms the feasibility of using a combination of teacher developed and commercially available materials to increase the use of computers in teaching mathematics.

Acknowledgements

The writer wishes to express her appreciation to Dr. Thomas E. Kieren who supervised the study on which this report is based. He initiated stimulating discussions on the conceptual basis of the project and provided guidance on the interpretation of the results. The comments and suggestions of the other members of the thesis committee, Dr. Alton T. Olson and Dr. Joan E. Worth, contributed in important ways to the completion of the project. Their assistance, which came at a crucial stage in the preparation of the report, is very much appreciated.

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Over a two-year period, the staff and students of the Department of Secondary Education provided an intellectually challenging and supportive environment. The contributions which they have made to my professional development are greatly appreciated.

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I. INTRODUCTION

Educators have always been faced with a two-fold task: (a) they are required to pass on knowledge from one generation to the next, and (b) they are required to socialize their students so that they will be able to cope with the demands of society. Neither of these tasks remains constant for our knowledge of the world around us is continually changing and, at the same time, society is also changing. Over the years educators have been able to rely on the traditional methods for going about their duties; however, in today's complex society this is no longer possible. The educators of the present and future must learn to adapt to the technological changes occuring in society and in the schools.

A. CHALLENGE OF COMPUTERS

In society today the "most significant technological achievement presently affecting the quality of life is the computer" (Braun, 1977, p.3). The computer has invaded the business world and, with the creation of micro-computers designed for personal use, the invasion has entered our homes. As Evans observes, the end to this invasion is nowhere in sight for

> during the eighties computers will become cheap and common rather than rare and expensive. Because they are small, they use up little raw material and can be produced in very large numbers. Because they

> have no moving parts (other than electrons), they are easy to mass produce. They cost little more than the raw materials of which they are made, and they have the advantage of being extremely reliable. (Evans, 1982, pp.15-16)

These developments raise a number of questions for educators. As computers begin to permeate all aspects of life, is it possible for educational systems to remain unaffected? If educators feel it is impossible to ignore the introduction of computers into schools without depriving students of valuable knowledge and the abilities required to function in today's technical society, then they are challenged as never before to revise, adapt and change current educational practices. The key, however, is the manner in which educators face this challenge. Thus far several approaches have been used in an attempt to integrate computers into the field of education.

During the sixties, emphasis was placed on the computer's ability to

> automate the teaching-learning process through the so-called 'tutorial mode' [in] an attempt to increase the productivity of teaching [and] decrease the number of teachers. (Hebenstreit, 1983, p.3)

This particular approach envisioned a day when education would be totally individualized, when students would be able to work at their own pace, explore their own areas of interest, and relate to material in a manner appropriate to their own learning styles. Critics of this approach, and there were many, envisioned a school in which students would sit before a computer terminal for hours at a time. In this type of situation very little, if any, human interaction might occur. A second, complementary approach, promoted the use of the computer for drill and practice. Here the intent was to relieve teachers of the responsibility for reviewing, practicing and testing previously learned skills, and thus to allow teachers time to focus on the responsibility of presenting material and assisting those students experiencing difficulty. The implementation of both of these approaches relied upon two key factors: (a) access to large computer systems, PLATO and TICCIT are two such examples, and (b) the availability of a large bank of software programs designed for all grade levels and all subject matter. As Critchfield reports, this approach "thrived in a situation where computers were justifiable only as cost-savers [and] the technology of the 1960s suggested this use of computers" (Critchfield, 1979, p.18).

During the seventies, the involvement of computers in education remained relatively unchanged. However, with the development of the microcomputer, the viability of the endeavor increased as the formidable cost of computers began to decrease. School systems quickly began to see their dreams of owning a computer become a reality; newly acquired machines were first put to use in the areas of administration and enrichment programs for students.

In summary, the first two decades of computer implementation in education focused on ways to improve the efficiency of teaching and learning. The major research question addressed during this period was:

> Can computers be used in education to do those things that we have been doing reasonably well? (Bell, 1978, p.430)

Bell reports that research studies answered the question with an emphatic "Yes."

In the eighties new approaches toward the use of computers in education are being brought to the forefront. As Bohrer observes:

> It is not sufficient to look through the curriculum and choose topics that can be "computerized." Instead, the computer should [be used] to enhance or expand a particular topic. (Bohrer, 1981, p.3)

To this end, educators are being encouraged to exploit fully the power of the computer so that new areas of interest, previously unavailable because of insufficient time or knowledge, may now be explored.

For educational researchers, the new approaches of the eighties require that their attentions be refocused toward questions such as:

> How can we use computers in mathematics classrooms to do better those things that are NOT presently being done very well? (Bell, 1978, p.430)

Questions such as this challenge educators to look at how computers can be used to complement the existing curriculum and current teaching methods, and to "search for creative ways to use the microcomputer to help us improve our teaching" (O'Daffer, 1983). The result of this change in approach and direction in research has been twofold: (a) it has resulted in looking for ways in which the computer can be used "by the learner as a tool" (Kearsley & Hunter, 1983,

p. 12-13), and (b) it has also resulted in focusing upon how teachers may benefit from using the new technology in their classrooms.

The intent of this study was to examine how the computer can best aid teachers and students in the teaching and learning of one unit of the High School Mathematics program currently in use in Alberta. The specific approach was to focus on how the computer may be used as a "facilitator of learning rather than the source of learning" (Wares, 1982, p.8).

B. PURPOSE OF THE STUDY

The specific purpose of this study was to design and test classroom materials for a unit on quadratic functions in which computer activities were integrated wherever appropriate, to teach the unit in an actual school setting, and to seek answers to a number of questions. The following questions guided the development of the study:

1. Does the use of the microcomputer and the integration of computer programming into the curriculum promote a relational understanding of mathematics?
2. What are the students' reactions to microcomputers and computer programming?
3. What is the teacher's reaction to the use of microcomputers and the inclusion of computer programming in the curriculum?

A general assessment of the instructional materials which were developed was implicit in the design of the study.

C. SIGNIFICANCE OF THE PROBLEM

A new dimension was added to the area of computing with the invention of the integrated circuit, the microprocessor and other related advancements in technology. The relatively affordable cost of personal computers has made it clear to see that computers are having a major impact on our lives:

> The word is out. On the main street, on the farm, even in the home, everyone knows that the fantastic power of the computer is now available -- even to schools. In spite of tight money, funding agencies are buying microcomputers for schools. (Wiederanders, 1982, p.7)

In the results of a survey conducted in early 1983, 63.4 percent of the schools in Alberta were reported as having one or more microcomputers. Even more surprising is the fact that:

> In the two year period from the spring of 1981 to the spring of 1983, the number of microcomputers has jumped from 265 to 3535, an increase of 1245 percent. (Alberta Education, 1983, p.18)

The recent government endorsement of the Apple II+ microcomputer for classroom use, and the success of computer conferences sponsored by the Alberta Teachers' Association Computer Council (ATACC) and the Alberta Society for Computers in Education (ASCE), may indicate that schools are acquiring microcomputers faster than had been anticipated. A recent statement by the Alberta Minister of Education, reported in the ATACC Newsletter, may be more accurate than many educators may wish to acknowledge. He said:

> I believe that the next five to ten years will see a saturation of computers throughout the educational system. (ATACC, 1983, p.5)

Clearly, computers are becoming a very real presence in today's classrooms.

As the computer begins to "saturate the system," educators are faced with the difficult task of determining the proper use of this new technology in the school setting. Nowhere is this challenge more evident than in the area of mathematics education. In their Agenda for Action, the National Council of Teachers of Mathematics strongly recommended that mathematics programs should "take full advantage of the power of calculators and computers at all grade levels" (NCTM, 1980, p.1). Specifically, the Council proposed the inclusion of electronic tools in the core mathematics curriculum and suggested that curricular materials which

> integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available. (NCTM, 1980, p.9)

Similar ideas have been presented by other mathematics educators (Wiederanders, 1982; Bodelier, 1978; Spencer and Baskin, 1981) who believe that the most effective introduction of the microcomputer would be as a complement to the existing curriculum. The technology, in this respect, would be used to aid teachers in explaining difficult concepts, to aid students in the understanding of algorithmic processes, to promote problem solving, and to illustrate significant applications of mathematics (Bell, 1978, p.431). In general, its purpose would be to enhance the curriculum as it now stands. The acceptance of this proposal requires that educators familiarize themselves with the new technology, design appropriate materials and uses for the microcomputer in the classroom, and evaluate the results of each endeavor.

While teacher preparation and inservice programs should be used to aid teachers in familiarization with the new technology, research can provide assistance in the design and evaluation of appropriate materials and uses for microcomputers in the classroom. The orientation of the present study was in the direction of this goal.

D. DELIMITATIONS

1. The study was restricted to the Mathematics 20 Unit on Quadratic Functions as prescribed by the Alberta Department of Education and to the elective computer literacy unit.

2. The study was conducted with one class of Honors students.

E. LIMITATIONS

1. The outcomes were dependent, in part, upon the classroom practices of a selected teacher and a particular class who worked with the researcher using carefully designed activities.

2. The study focused only on the short term effects of the introduction of microcomputers and computer programming; longitudinal effects were not be considered.

3. The programming language learned and applied by the students was BASIC.

F. OVERVIEW OF THE REPORT

In Chapter Two the theoretical background on which the study was based is outlined and the intent of the study is summarized. Chapter Three describes the design of the study as well as the basis for specific decisions. The chapter concludes with an explanation of the materials developed for testing. In Chapter Four the results of the research project are presented and the observations are related to the theories outlined in the second chapter. Chapter Five contains a summary of the project, the conclusions and suggestions for further research.

II. REVIEW OF RELATED LITERATURE

The purpose of this chapter is to build a theoretical framework for the primary research question: Does the use of the microcomputer and the integration of computer programming into the curriculum promote a relational understanding of mathematics? What follows is a discussion of the term understanding from various psychological perspectives and a review of the literature concerning how these theories are applied in mathematics education. The chapter concludes by relating the intent of this study to the preceding discussion.

A. LEARNING AND UNDERSTANDING

A familiar maxim states that "knowledge is power." However, many educators have suggested that knowledge in and of itself is not power but, rather, that power stems from the ability to use and apply knowledge. The recognition of this distinction leads to the identification of two distinguishable forms of learning: learning which focuses on rote memorization of facts, or habit learning, and learning which involves understanding. Richard Skemp, in his book The Psychology of Learning Mathematics, elaborates on these two forms of learning. He states:

> The former [form of learning] can be replicated in the laboratory rat or pigeon, and for various reasons (such as the greater degree of experimental control which is possible) contemporary

> psychologists have long seemed to prefer this kind of learning. ... The latter kind of learning is that in which man most excels, and in which he most differs from all other species. (Skemp, 1971, p.15).

The focus of psychological research on rote or habit learning is not surprising considering the strong historical influence of behavioral psychology on learning theories. With the growing strength of cognitive psychology and information processing theories, together with the gradual blurring of the boundaries separating cognitive and behavioral psychology, research is shifting toward the second type of learning, namely, learning which involves understanding.

The area of disagreement between various branches of psychology has not focused on the importance of understanding because

> the basic fact that understanding promotes learning and retention has never been doubted. (Bower & Hilgard, 1981, p. 318).

Rather, the area of disagreement has focused on the interpretation of what it means to understand and the educational practices related to the various interpretations of understanding. The differences are evident in the interpretations of the behaviorist and cognitive branches of psychology.

Behaviorist Interpretations

To associate the term understanding with Stimulus-Response (S-R) theorists seems to be contradictory.

However, Thorndike, Guthrie and Skinner all acknowledged and interpreted the term within their own theoretical frameworks. While Thorndike did not deny insight and understanding in human beings, neither was he awed by it. He believed that understanding could be interpreted according to the association laws he had formulated and applied in other situations (Bower & Hilgard, 1981). Understanding and assimilation, in this framework, were said to be synonymous as both terms referred to an individual's ability to respond to a new situation of a similar nature. The response to the new situation was formed by adapting the response given in the previous situation. Therefore, according to Thorndike, understanding was said to "grow out of earlier habits" and was best fostered by "teaching students the many connections relevant to a problem" (Bower & Hilgard, 1981, pp. 47-48). A similar view of understanding was advocated by Guthrie; however, he placed less emphasis than did Thorndike on this form of learning.

While Skinner himself rarely addressed the idea of learning with understanding, other behaviorist psychologists did use and define the term. According to Bower and Hilgard (1981), for these psychologists understanding was dependent upon verbal description since properly labelling a situation increased the probability of activating the appropriate response sequence. The "emergence of the solution was said to be explained on the basis of (1) similarity of the present problem to one solved earlier, or (2) the simplicity of the problem" (Bower & Hilgard, 1981, p.197). Following this perspective, problem solving is viewed as the manipulation of variables to evoke a response. Furthermore, the necessary skills can be taught to students.

In summary, a behavioristic view of understanding centers on the subject's ability to see similarities between a new situation and a situation previously experienced, and the ability to activate the appropriate S-R chain to arrive at a response. It would probably be safe to say that the speed at which one can identify the similarities between situations and activate the correct S-R chain greatly influences whether one's actions are viewed as occuring with or without understanding. That is to say, the faster the chain is identified and activated, the more likely it will be that the subject is said to respond with understanding or insight.

Cognitive Interpretations

Cognitive psychology, with its emphasis on behavior being determined by thought processes rather than by automatic responses, focuses on a view of understanding that involves "grasping the principles underlying a sequence of episodes" (Bower & Hilgard, 1981, p. 317). This view of psychology and interpretation of understanding was brought to the forefront of psychological investigation with Kohler's studies of apes in problem solving situations. Kohler reported that many of the problems presented to the apes were solved in ways that could only be said to rely on insight because of (a) the suddenness of the solution, (b) the transferability or adaptability of the solution, and (c) the extent to which understanding of the total situation was involved in the solution (Gardiner, 1980). Kohler's research, and studies conducted later by Wertheimer involving students generalizing the formula for the area of a rectangle to a formula for finding the area of a parallelogram, contributed greatly to the formulation of the Gestalt position on what it means to understand.

From a Gestalt position understanding is interpreted as involving

> the perceiving of relationships, awareness of relationships between parts and whole, [and] of means to consequences.(Bower & Hilgard, 1981, p.322)

This holistic approach is emphasized in problem solving situations which are said to be approached

> sensibly, structurally, [and] organically rather than mechanically, stupidly, or by the running off of prior habits. (Bower & Hilgard, 1981, p.323)

Therefore, the Gestalt position exists in direct opposition to the associationist approach.

Information Processing

Research into information processing theories and artificial intelligence, which have been viewed as a part of theoretical psychology, also provides a perspective on understanding. Both areas have as their basic premise the agreement that knowledge regarding how people process information can be gained by simulating cognitive processes or behavioral actions on a computer. According to this perspective

> a machine which accurately simulates relevant aspects of some organism's behavior indeed constitutes a genuine explanation of that behavior. (Bower & Hilgard, 1981, p. 354)

Research into what it means to understand in these areas has followed two distinctly different approaches:

1. to understand an item of knowledge refers to the ability to use the knowledge in appropriate situations; and,
2. to understand an item of knowledge refers to the ability to make inferences about the knowledge or to extend it.

Understanding as the ability to use knowledge has been the focus of semantic-based parsing systems which attempt to interpret sentences and respond to questions based on information contained in the sentence.

Most research in this area has centered around the development of understanding programs most notably SAM, developed by Schank (1972), and MERLIN, developed by Moore and Newell (1974). It is in these programs that the breaking down of the barriers between associationists and cognitivists are most evident. The understanding program SAM, which defines understanding in terms of the ability to make inferences, relies on "scripts" to fill in missing connections and to make inferences. Scripts have been defined as stereotyped sequences of events or activities that refer specifically to a

> particular goal-plan sequence [that] has been used repeatedly by a person in a standard situation. Schank and Abelson suppose that it becomes routinized into a conventional activity. (Bower & Hilgard, 1981, p. 408)

This definition and the behaviorist's definitions of S-R chains and understanding are indeed very similar.

The educational implications and practices advocated by the cognitive psychologists have particular relevance for the development of understanding in the teaching of mathematics.

B. UNDERSTANDING THEORIES AND MATHEMATICS EDUCATION

Discussions concerning what it means to understand mathematics from a cognitive viewpoint have been greatly influenced by information processing theories and research into the understanding of language. These discussions are based on a "concept of understanding [which] refers to the construction of a representation of some information" (Greeno, 1978, p. 263). Similar definitions have been advocated by Henry van Engen who spoke of understanding as a "process of integrating concepts" or "fitting it [an idea] into a conceptual structure already in the pupil's possession" (van Engen, 1953, pp. 76-77), by Margenau (1961) who viewed mathematics as a network of constructs "or general ways of knowing possessed and developed by the learner" (Kieren, 1978, p.77), and by Richard Skemp (1971) who referred to the representation or conceptual structure as a schema. Resnick and Ford (1981) also support this definition and have diagramatically shown how schemas are formed and assimilated (see Figures 1, 2, and 3).

The schemas in Figure 1 and Figure 2 represent differences in understanding of the relationship between multiplication and division. In Figure 1, the separation of the schema indicates a lack of understanding of the relationship between the two operations. The joining and simplification of the schema in Figure 2 indicates an understanding of the inverse relationship between multiplication and division. Figure 3 presents a schema which suggests a well-developed understanding of the operations of addition, subtraction, multiplication and division as well as of the relationships among the four operations.

Measuring Understanding

Understanding is not a simple process. Although understanding may vary in degree from person to person and situation to situation, it is possible to measure an individual's understanding of a particular event or item of knowledge. Greeno (1978) has identified three criteria which may be used to judge the degree of understanding: coherence, connectedness and correspondence. Although Greeno's criteria were established for evaluating an individual's understanding of language, they may also be applied in evaluating the understanding of mathematics. Resnick and

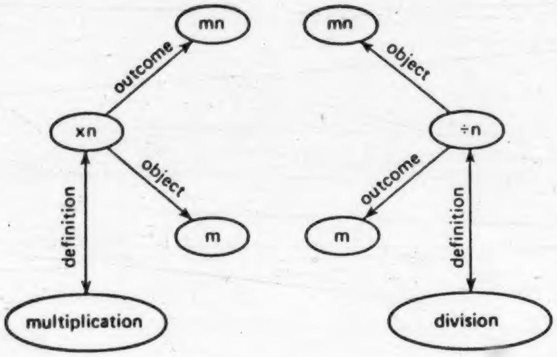
mn
outcome
xn
object
m
definition
multiplication
mn
object
÷n
outcome
m
definition
division

Figure 1. Schema representing nonunderstanding of relationship between two operations (Source:Resnick & Ford, 1981, p. 201)

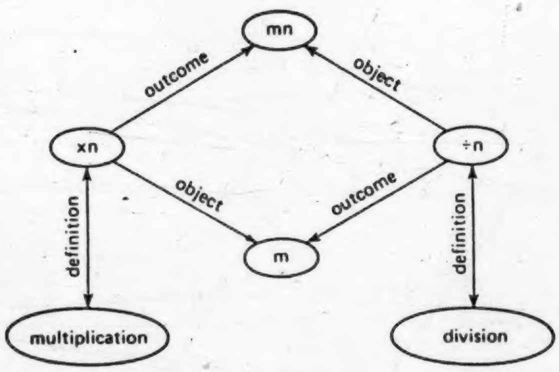
mn
outcome
object
xn
÷n
object
outcome
m
definition
definition
multiplication
division

Figure 2. Schema representing understanding of relationship between two operations (Source: Resnick & Ford, 1981, p. 202)

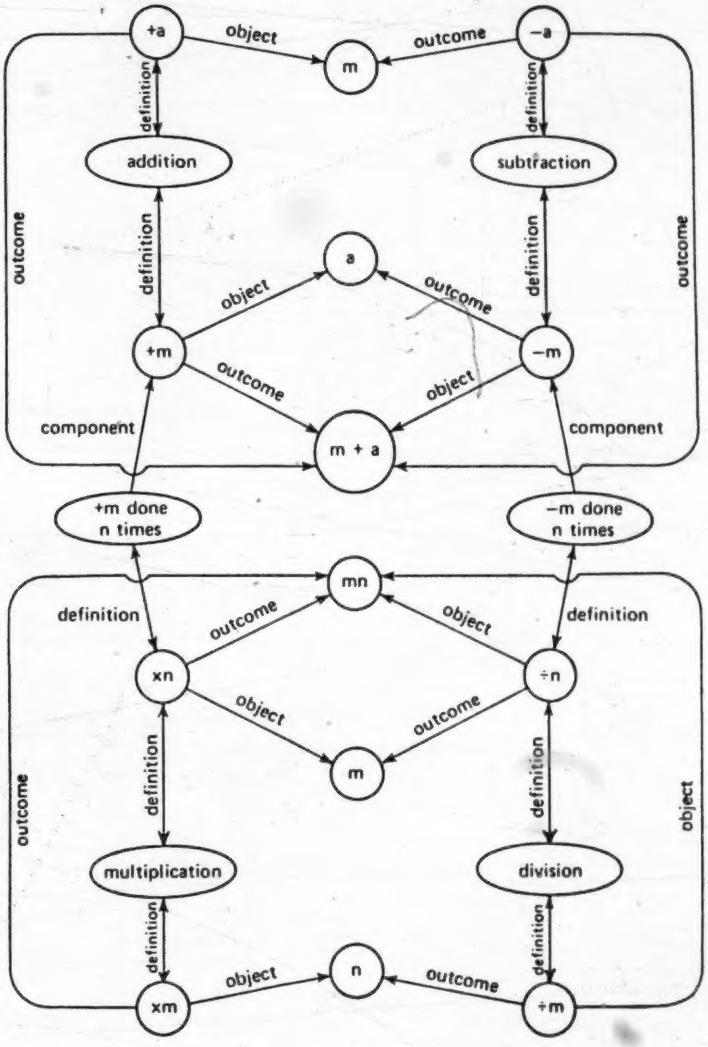
+a
object
m
outcome
−a
definition
definition
addition
subtraction
outcome
definition
a
definition
outcome
object
outcome
+m
outcome
object
−m
component
m + a
component
+m done n times
−m done n times
mn
definition
outcome
object
definition
xn
÷n
object
outcome
m
outcome
definition
definition
object
multiplication
division
definition
definition
xm
object
n
outcome
÷m

Figure 3. Schema representing understanding of relationships among basic operations (Source: Resnick & Ford, 1981, p. 204)

Ford, however, have advocated that the first criterion be changed to "internal integration of the representation" (Resnick & Ford, 1981, p. 206) since Greeno's original term coherence did not imply logical coherence. The criteria for measuring an individual's understanding of mathematics then becomes (1) internal integration of the representation, (2) degree of connectedness of the information to previous knowledge, and (3) correspondence of the individual's schema to the material that is to be understood.

To assess an individual's internal integration of the representation or schema, one attempts to determine "the extent to which concepts are associated with each other in rich yet orderly ways" (Resnick & Ford, 1981, p.207). This is illustrated in the difference between Figure 1 and Figure 2. Such assessments have been conducted using timed free-recall experiments and are based on semantic network theory. The basic premise has been that integrated schema consist of very few organizing concepts linked to a larger number of other concepts. During a free recall experiment, patterns of information retrieval will be exhibited in which experts retrieve information in chunks and novices retrieve information more randomly. Resnick and Ford report that this was successfully demonstrated in a series of experiments involving physics teachers and their students; however, they also reported that:

> Larkin's method allows one to conclude only that information is chunked and it reveals nothing about which items are chunked together. (Resnick & Ford, 1981, p. 208)

In addition, Larkin's method does not allow the researcher to discover the correctness of the links. This aspect will be examined in the discussion of the third criterion for measuring understanding.

The second criterion for measuring understanding, namely, the degree of connectedness of the information to an individual's previous knowledge, refers to the ability to apply knowledge in new situations. Most of the research in this area has focused on interviewing and observing students in problem solving situations and then inferring information about their schemas from their performance. The method has been successful, and often information regarding the first and third criteria can be gained from watching students perform. However, as Resnick and Ford point out, more appropriate ways of inferring knowledge of people's schemas must be developed.

The third and final criterion is concerned with the correspondence of an individual's schema to the material that is to be understood. This criterion refers to how well an individual's schema matches that of an expert. A mathematically correct schema of the four basic operations is shown in Figure 3. Assessment of this criterion has been conducted through word association, graph building and card sorting tests. These instruments were used by Shavelson (1972) to show how the cluster patterns of physics students, as they progressed through a course, moved closer and closer to the cluster pattern displayed in the instructional material. A later study by Thro (1978) resulted in similar findings and also showed how the cluster patterns of successful students were more closely matched to the instructor's cluster patterns than were those of unsuccessful students.

Although these criteria do not provide a definite means of assessing understanding in mathematics or any other subject, they do provide a starting point for measuring a very elusive process.

Instrumental vs. Relational Understanding

Many of the points mentioned by Greeno (1978), and by Resnick and Ford (1981) relate to earlier work done by Richard Skemp. The main area of difference is that while Greeno, and Resnick and Ford, were concerned with determining how understanding could be assessed, Skemp focused more on defining understanding. Skemp (1976) was able to distinguish two meanings of the word understanding: instrumental understanding and relational understanding.

Instrumental understanding, as defined by Skemp, refers to the possession of rules without reasons or a knowing "how to" but not "why." Skemp contended that this is the type of understanding that has received the greatest attention in schools, especially in teaching mathematics. To support this contention, Skemp asks us to consider how many students have learned to divide fractions by "turning the second one upside down and multiplying" without understanding how the rule was generated. That such mathematics has been the focus of teaching need not be elaborated further. Skemp points out that the reason why we focus on instrumental understanding is that its rewards are immediate and easily gained. Instrumental understanding provides us with a quick and efficient algorithm for completing problems that, in turn, generates a page of correct answers. The difficulty with instrumental understanding is that it "necessitates memorizing which problems a method works for and which [it does] not, and also learning a different method for each new class of problems" (Skemp, 1971, p.23). Instrumental understanding fails to aid the student with internal integration of the schemas and with connecting new information to previous knowledge -- two important criteria for measuring understanding according to Greeno (1978), and Resnick and Ford (1981).

The second type of understanding identified by Skemp, relational understanding, refers to both knowing what to do and why. The advantages of relational understanding include the following:

1. relational understanding of mathematics is more adaptable;
2. while harder to learn initially, relational understanding proves to be easier to remember;
3. relational understanding may become a goal in itself; and,
4. relational schemas are organic rather than

mechanical in nature.

In other words, relational understanding allows an individual to construct an adaptable, functional, and "correct" schema in tune with the criteria presented by Greeno (1978), and by Resnick and Ford (1981).

Richard Skemp's views, published in an article entitled "Relational Understanding and Instrumental Understanding" in Mathematics Teaching, precipitated a rather lengthy debate revolving around the many different types of understanding that could be exhibited in mathematics. The debate is summarized in Table 1.

While many mathematics educators have argued that Skemp's view of understanding should be expanded, the important point is not the number of classifications but the acknowledgement that there are various forms of understanding that can be "taught." Educators must remember that through the formation of appropriate schemas students' comprehension of the subject matter under discussion will improve as schemas allow for the "integration of existing knowledge and they [provide for] the acquisition of new knowledge" (Skemp, 1971, p.39). On the other hand, the formation of inappropriate schemas makes assimilation of new information difficult, if not impossible, and thereby decreases students' ability to understand.

Table 1

Summary of Differing Interpretations of Understanding

Skemp (1976)	Byers and Herscovics (1977)	Backhouse (1978)	Godfrey (1978)
Instrumental Understanding Possession of rules and the ability to use them	Instrumental Understanding The ability to apply the appropriate rule without being able to explain why it works	Instrumental Understanding Possession of rules and the ability to use them	Technical Understanding Ethological Understanding Attempting to deal with mathematical symbols with no knowledge of their meaning
Relational Understanding Knowing what to do and why	Relational Understanding The ability to deduce certain rules from more general mathematical relationships	Relational Understanding Knowing what to do and why	Relational Understanding The ability to deduce certain rules from more general mathematical relationships
	Intuitive Understanding The ability to solve a problem without prior analysis		Intuitive Understanding The ability to solve a problem without prior analysis
	Formal Understanding The ability to connect mathematical symbolism with relevant ideas and to combine the ideas to create chains of logical reasoning		Formal Understanding The ability to connect mathematical symbolism with relevant ideas and to combine the ideas to create chains of logical reasoning

Buxton (1978)	Skemp (1979)	Skemp (1982)
Rote Purely instrumental understanding in which information is imprinted on the brain	Instrumental Understanding The ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works	Instrumental Understanding The ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works
Observational Perceive a relation or pattern which allows one to check an answer or make an educated guess	Relational Understanding The ability to deduce specific rules or procedures from more general mathematical relationships	Relational Understanding The ability to deduce specific rules or procedures from more general mathematical relationships
Insight Understand how and why		
Formal Proof for the mathematician (This level is appropriate only after the other levels have been reached)	Formal (Logical) Understanding The ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning	Symbolic Understanding A mutual assimilation between a symbol system and a conceptual structure, dominated by the conceptual structure

In summary, it has been suggested that a "major goal of mathematics instruction should be the acquisition of 'well-structured' knowledge about mathematics" (Resnick & Ford, 1981, p.205). In order to achieve this goal, educators should focus attention on how mathematics programs can aid students in the development of appropriate schemas. The extent to which this goal is presently being met can be determined through a review of current mathematics programs.

C. CRITIQUES OF MATHEMATICS PROGRAMS

In 1982 the National Assessment of Educational Progress (NEA) completed its third survey of 9, 13, and 17 year olds enrolled in mathematics programs throughout the United States. The findings, reported at the 1983 Annual Meeting of the NCTM, showed that improvements had been made in those areas that "assess things most easily taught and learned by rote" but that "performance on nonroutine problems and on problem solving continued to be exceptionally low" (NCTM, 1983, p.2). These results were in agreement with a previous study conducted by the International Association for the Evaluation of Educational Achievement (IEA). The IEA study found that algorithmic approaches to mathematics tended to be heavily emphasized, particularly in the area of computation, with the result that students, in general, possessed a relatively low understanding of mathematics. Robitaille summarized the IEA findings by stating that "in general, the results show that achievement decreases as cognitive level increases" (Robitaille, 1983, p.7). The results of the NEA and IEA studies support the views of critics of mathematics programs and are in agreement with the results of other studies conducted on a much smaller scale.

For many years now, mathematics educators have been critical of the mathematics programs presently in use. They have advocated that "much of what is being taught and learned under the description of 'modern mathematics' is being taught and learned just as instrumentally as were the syllabi which have been replaced" (Skemp, 1976, p. 22). The net result for students is that mathematics becomes the "manipulation of symbols having little or no meaning attached, according to a number of rote-memorized rules" (Skemp, 1971, p.31).

In a study designed to assess understanding as an end-product of learning, Saad (1960) tested 3,324 fourth and fifth year students in England, in either arithmetic, algebra or geometry. The results of this study verified Skemp's contentions when it was found that:

> The teaching methods to which children have been subjected tend to develop not meaningful responses but fixations, so that a child may know a formula without knowing how this formula has developed or why it is always valid. (Saad, 1960, p.119).

Using a different approach to the question of whether or not school mathematics programs promote understanding, Bell (1970) investigated mathematics textbooks in use prior to 1960 and those in use since 1960. He came to

> the same 'gloomy conclusion' that Thorndike reached in a similar study in the 1920s: "In general, teachers and youngsters who seek understanding of ... mathematics will get little help from school algebra books." (Travers, Pikaart, Suydam, & Runion, 1977, p.7)

Conclusions such as these confirm Skemp's contention that school mathematics programs promote instrumental understanding to the detriment of relational understanding. The net result is that mathematics educators are now faced with the difficult task of designing and implementing new approaches to the teaching and learning of mathematics. Specifically, mathematics educators should be concerned with the development of a curriculum which will aid students in the formation of appropriate schemas and thereby promote relational understanding. This particular point was addressed by Hebenstreit at the IFIP Working Conference on "Involving Micros in Education." He stated:

> The transmission of knowledge should be systematically designed so that pupils can broaden, deepen and improve their model-building aptitudes rather than more or less completely reproduce some subset of knowledge, which they have tried to memorize as best they can. (Hebenstreit, 1983, p. 20)

To achieve this end, he promoted the use of computers in education for

> if a computer is regarded as a device for manipulating symbols according to purely syntactic rules and programming languages merely as a set of methods for elucidating these rules, clearly the fundamental problem in computer science is neither the computer nor the machine language, but the formulation of an actual situation in purely syntactic terms, i.e., the construction of a model. (Hebenstreit, 1983, p.21)

The ways in which computers can assist in the construction of appropriate models is of interest to both researchers and mathematics educators.

D. COMPUTERS AS MATHEMATICAL ENTITIES

The views expressed by Hebenstreit are not entirely new; educators from Dewey to Piaget have emphasized that "in order to understand a concept, students need to take an active role" (Soloway, Lochhead & Clement, 1981, p.1). Recently the most adamant and vocal proponent of this approach, particularly in the area of computer integration, has been Seymour Papert (1980, 1981) whose views have developed from his work with Piaget and the LOGO project at MIT.

From his work with Piaget in Switzerland, Papert came to realize that "children [are] the active builders of their own intellectual structures" and that in the building of these structures "children appropriate [for] their own use materials they find about them, most saliently the models and metaphors suggested by the surrounding culture" (Papert, 1981, p.3). The difficulties children experience in mathematics, Papert feels, occur because "our education culture gives mathematics learners scarce resources for making sense of what they are learning" (Papert, 1980, p.47). Through the use of computers, however, Papert sees an opportunity to transform a sterile culture into a mathematically rich culture for "the metaphor of computer as a mathematics speaking entity puts the learner in a qualitatively new kind of relationship to an important domain of knowledge" (Papert, 1981, p.3). Specifically, Papert sees the transformation arising in the following manner:

> When a child learns to program, the process of learning is transformed. It becomes more active and self directed. The knowledge is acquired for a recognizable personal purpose. The child does something with it. The new knowledge is a source of power and is experienced as such from the moment it begins to form in the child's mind. (Papert, 1981, p. 3-4).

While Papert sees the "cultural" transformation occurring primarily through the use of the LOGO language and environment, many other educators feel it is the algorithmic process itself that places students in a fundamentally new position in regard to mathematics.

Expanding upon earlier conceptions of the problem solving cycle, Kieren (1978) distinguished three stages of development necessary for the construction of mathematical ideas. The stages he identified are: exploration of elements, formal control, and advanced exploration. The first stage, exploration of elements, refers to the period of time during which students familiarize themselves with the essential components of an idea. Following the exploration stage, an individual is able to manipulate the symbols and language associated with a concept in the appropriate manner. It is during this second stage that the relationship explored in the first stage becomes formalized. Lastly, the advanced exploration stage refers to the period of time during which a "construct is tied to other constructs through applications, generalizations and problem solving" (Kieren, 1978, p.78). While informatics experiences are valuable at all three stages of the cycle, Kieren feels the "most profound contribution of informatics to mathematics curriculum is its approach to symbolism" (Kieren, 1978, p. 81).

For secondary students, the power of mathematics "comes from the use of symbols as an analytic tool"; however, for many students learning mathematics it is this "very notion, symbolics, which presents such great difficulty" (Kieren, 1978, p. 81). Proponents of an informatics approach to mathematics advocate that it

> offers a means to symbolic control over various mathematical constructs [because] an algorithm for a machine is a bridge between the concrete stage of thought and the more formal stages of thought. (Kieren, 1978, p. 81, 84)

In addition, as Skemp observes, often "the mere act of communicating our ideas seems to help clarify them, for, in doing so, we have to attach them to words (or other symbols), which makes them more conscious" (Skemp, 1971, p. 121). It is through such involvement, Skemp contends, that individuals develop the ability to reflect upon their own schemas. In this regard, an algorithm may be said to reflect an understanding of the mathematical problem under investigation and of the symbolism involved. Through the development of an algorithm, individuals are offered the opportunity to reflect upon their own schemas.

E. RELATED STUDIES

Research into the use of computers in mathematics instruction began in the late sixties as schools acquired remote terminals, purchased their own mini-computers or shuttled programs and output between schools and computer centers. Generally, the studies have been conducted from a quantitative perspective, and most have been presented as dissertations. The tendency has been to focus on secondary and post-secondary algebra and calculus courses to determine the effects which computer-oriented mathematics has upon achievement, problem solving ability, reasoning ability, and attitudes. The majority of results have seemed to indicate no significant difference between treatment groups.

The effects of computer oriented techniques on achievement have received the most emphasis. The overall results seem to be inconclusive. A number of researchers (Kieren, 1969; Marchand, 1974; Payne, 1980; Krull, 1980) have reported a significant difference in favor of the computer treatment while others have found that the use of the computer neither improved nor interfered with achievement (Hoffman, 1971; Mandelbaum, 1974; DeBoer, 1974; Basil, 1974; Strawn, 1975; Deloatch, 1978; Sears, 1978; Chuchat, 1985). In the areas of problem solving and reasoning ability the results have not been as divided; results have indicated that computer programming and flowcharting do facilitate development in these areas (Hatfield, 1970; Foster, 1973; Milner, 1973; Ronan, 1971;

Wilkinson, 1973; Andreoli, 1976). With regard to effects upon attitudes, the results are again inconclusive. Studies by Milner (1973), Strawn (1975), Deloatch (1978), Deblassio (1978), and Chuchat (1985) found the use of the computer and the inclusion of computer programming in mathematics contributed favorably to attitude development. Similar studies by Basil (1974), DeBoer (1974), Mandelbaum (1974), and Krull (1980) concluded the opposite.

While these studies seem to indicate that computer augmented mathematics does not greatly enhance the overt, easily measurable areas of mathematics, many of these same studies have indicated that a more subtle outcome may result from implementing this new approach to mathematics teaching. In one study of undergraduate non-mathematics majors, an algorithmic approach to problem solving resulted in "strong justification for incorporating computer oriented techniques into a modern syllabus" (Marchand, 1974, p.2888). Students were: a) better able to structure their solutions to problems using algorithms, flowcharting, and iteration; b) showed significant improvement in the development of knowledge and skills; and c) were motivated to re-study material from earlier school years. Wells (1981), in a similar study involving high school students, also concluded that computer programming is an effective way to evoke problem solving processes. Specifically, Wells based her conclusion on the observation that "persistence and looking back techniques [were] widely used in programming" (Wells, 1981, p.2010A).

Similar subtle positive outcomes were shown in other studies. Ducharme (1974) studied sophmore level college students in a computer-oriented linear algebra course and found that the use of flowcharts improved the students' understanding of mathematics. He also found that flowcharts were indispensible for the understanding and computer implementation of algorithms. In a fifteen-week study of grade five students involved with interactive programming in LOGO, Milner (1973) assessed the value and effectiveness of computer programming in the learning of mathematics. He concluded that it was "an effective learning resource in terms of both cognitive and affective considerations" (Milner, 1973, p. 4184-A). In a separate study also involving LOGO, Seidman (1981) reported that programming in LOGO significantly enhanced logical reasoning ability under certain conditions.

In studies concerned with the effect computer use has on classroom dynamics, Cox and Berger (1981) found that student grouping at computer terminals was related to performance and recommended that student grouping should not be left to chance. Specifically, Cox and Berger suggested that groups should be limited to "two or three members [since with] larger groups one or two students were often left out" (Cox & Berger, 1981, p.29). In contrast, Dugdale (1982) concluded that grouping was not a significant factor in student performance; students working in groups at computer terminals seemed to contribute on a more or less equal basis.

The three projects which most closely parallel this particular study are the Nottingham Programming in Mathematics Project (Hart, 1981) and recent studies conducted by Soloway, Lochhead and Clement (1981) and by Chuchat (1985). These studies, because of their relevance, will be explored in greater depth.

The main objective of the Nottingham Programming in Mathematics Project was to identify more clearly areas in which programming activities would aid students in their understanding of mathematics at the lower secondary level, and to "devise and test teaching materials that are directed towards achieving the resulting objectives" (Hart, 1981, p. 82). Specifically, the project focused on improving students' understanding of the concept of variable as this concept, from the results of recent Concepts in Secondary Mathematics and Science (CSMS) surveys, seems to be poorly understood. Programming was seen as an appropriate approach to increase understanding because it provides for:

(i) "a concrete embodiment of the concept or at least a means of approach that makes a concrete visualization possible" (Hart, 1981, p.83); and,

(ii) it contains the concept in activities which are meaningful to the students.

While the results of the study were not statistically significant, perhaps due to a small sample, they did seem to indicate that computer programming aids students in the development of the concept of variable. In addition to the students' improvement in written exercises, the most notable evidence was the increase in the students' performance on the following four questions which were included in both the pre-test and the post-test:

1. add 4 onto n + 5;
2. add 4 onto 3n;
3. multiply n + 4 by 4; and,
4. Cakes cost c pence each and buns cost b pence each. If I buy 4 cakes and 3 buns, what does 4c + 3b stand for?" (Hart, 1981, p.83)

Analysis of the students' responses showed that the success rate increased from 41 to 75 percent on the first question, 20 to 45 percent on the second question, 0 to 12 percent on the third question, and 4 to 16 percent on the last question. In conclusion, Hart stated:

> The relationship between research activity concerning pupil understanding and the use of programming certainly promises to be a very fruitful one. (Hart, 1981, p. 88)

The Soloway, Lochhead and Clement (1981) study focused upon students' ability to read and understand algebraic word problems embedded in a programming context as opposed to the ability to read and understand these problems in isolation. The specific concern of the researchers was that students seem to "have developed special purpose translation algorithms which work for many textbook problems, but which do not involve anything that could reasonably be called a semantic understanding of algebra" (Soloway, Lochhead & Clement, 1981, p. 3). This point was illustrated by first year college students who, when asked to write an equation expressing the relationship between two sets, wrote equations indicating the relative size of the two sets rather than equations expressing how one set could be transformed into the other. It was concluded that students viewed an equation as expressing a passive picture of the two sets rather than an active operation.

To further explore this area of mathematical understanding, Soloway, Lochhead and Clement (1981) conducted two investigations to determine if those students who were placed in an environment which allowed them to take a more active procedural view of equations exhibited a lower error rate on problems than students not placed in such an environment. The researchers felt that one clear candidate for such an environment would be the area of computer programming since a "computer program is a definite prescription for action; it is a set of commands which produce some result" (Soloway, Lochhead & Clement, 1981, p. 6). The results of the investigations showed that significantly more students could answer a word problem by writing a computer program than an algebraic equation. Furthermore, three times as many students could read and explain an equation embedded in a computer program as opposed to an equation which stood alone. The results supported the initial hypothesis that "programming does enhance problem solving because programming encourages the needed procedural view" (Soloway, Lochhead & Clement, 1981, p. 13). However, as indicated by the researchers, much more work needs to be done in this area to determine more specifically the cause/effect relationships involved.

In a study which closely parallels this particular project, Chuchat (1985) investigated the incorporation of computer programming into the Mathematics 20 curriculum. Specifically, the Chuchat study involved two units: a computer literacy unit which required students to learn how to program in BASIC, and a computer implementation unit in which students were required to apply their knolwledge to write computer programs to solve "typical" end of chapter exercises. The purpose of the study was to determine the benefit which the activities would have for students and their reactions to the experience. Results of the study indicated that students involved in the computer programming group required more attention and a greater amount of explanation from their teachers than did those who were not involved. A significant difference in favor of the computer group was found only in areas which required a considerable amount of programming. In areas which had few programming activities, the differences favored the control group. The most significant gains were observed in relation to student reactions and attitudes. The students in the computer group reported that they preferred to do mathematics which involved programming activities and also expressed a more positive attitude toward mathematics.

In summary, research results seem to have been inconclusive with regard to the effect computer oriented mathematics has on attitudes and achievement; however, benefits from this approach are being shown in the area of problem solving ability, reasoning ability, and understanding. Continued growth in this area of research would be enhanced by direct investigation of the effects which computer oriented mathematics instruction has upon these latter areas.

F. RELATIONSHIP TO PRESENT STUDY

The preceding discussion has focused on defining "understanding" from various psychological perspectives and in terms of mathematics education. An intent of this study was to determine whether a student's understanding of mathematics can be improved through the integration of computer programming into the curriculum. A basic assumption underlying the project was that a close relationship exists between the process of programming a computer and the processes involved in developing understanding of mathematical operations.

In forty short years computers have become a fact of life. While it often seems that computers are controlling our lives, in reality, computers are just an

> automatic device that is constructed in such a way that it can carry out any one of a number of well-defined basic information-processing operations. (Booth & Chein, 1974, p. 2)

The power of a computer emanates from the programs upon which it functions. In the process of programming a computer, the programmer describes a computational task as a sequence of well-defined steps commonly referred to as an algorithm. The programmer's most important task in programming a computer is to define the algorithm in a clear, exact, and unambiguous manner. To do so requires understanding.

Earlier in this chapter, understanding was defined from a variety of perspectives. In the discussion it was indicated that many mathematics educators (Greeno, van Engen, Margenau, Skemp, Resnick and Ford) regard understanding as the construction of a representation of some information. Furthermore, understanding is a process in which relationships must be shown and generated by the student. It has been suggested that in

> teaching for understanding we should consider the mental activities that a student must perform in order to understand and arrange exercises to provide practice in these activities. (Greeno, 1978, p. 267)

The premise upon which this study was based is that the algorithm required to program a computer may be equated to the student's representation of information or schema. Both theory and research suggest that the process of generating, testing and refining a schema improves a student's understanding.

In the present study, Greeno's criteria for measuring understanding and Skemp's definition of relational understanding were defined operationally as is indicated below.

Coherence. The term coherence was used to refer to the individual's internal integration of the subject matter under discussion. For the purpose of the study, coherence was used to analyze the completeness of the students' programs.

Connection. The term connection refers to the extent to which newly acquired knowledge can be integrated into existing schemas. In the study, connection was used to refer to the students' ability to relate correctly new and existing knowledge.

Correspondence. The term correspondence refers to the extent to which students' schemas agree with or correspond to the existing body of knowledge. Correspondence was used to analyze the "correctness" of a response.

Relational Understanding. Students who know both how to proceed and why to proceed in that manner were described as possessing a relational understanding of mathematics.

If schema organizing activities are required to program a computer properly, then the implication is that computer programming which is integrated into the curriculum may lead to increased understanding of mathematics. The operational definitions derived from Greeno and Skemp were used to explore this relationship in the present study.

III. DESIGN OF THE STUDY

The project which is the subject of this report consisted of three main parts: development of instructional materials, testing these materials in a classroom setting, and using the results of the test to answer a number of research questions. During the design phase of the study, decisions had to be made about the specific materials to be developed, the selection of a site for testing the materials, and procedures to be followed during the test.

A. MATERIALS DEVELOPMENT

In order to address the basic research questions, the materials to be developed had to have potential for computer programming to be integrated into the curriculum. In addition, the content had to provide for testing the extent to which the programming contributed to the development of a relational understanding of mathematics.

Topic and Grade Selection

Following a review of alternatives, the teaching of quadratic functions was selected for study. An understanding of the concepts and principles involved in the Mathematics 20 unit on quadratic functions plays an important role in a student's future success in high school mathematics since this unit is an introduction to many of the ideas upon which

Mathematics 30 and 31 are based. The decision to select Mathematics 20 students who were studying this unit was influenced by the following considerations:

1. the results of previous research in the area of integrating computers into the mathematics curriculum;
2. the provision in the Mathematics 20 curriculum for elective units; and,
3. the general feeling on the part of teachers that this unit was difficult for students.

In Alberta, Mathematics 20 is the matriculation mathematics course offered at the Grade 11 level.

The results of studies conducted in the late sixties by Hatfield (1970) and Gumm (1970) support the use of the computer with students of average or above average ability. A recommendation which flows from these studies is that

> the computer be introduced to the secondary college bound student in conjunction with the mathematics curriculum. (Gumm, 1970, p.3899A)

With respect to the use of computers specifically in the unit on quadratic functions, Kieren (1969) found a significant difference in achievement on a unit test on quadratic functions in favor of the computer group. He concluded that:

> The computer treatment appears to contribute most to the learning of complex processes, organization and relation of data, and infinite processes. (Kieren, 1968)

Comprehension of these areas in relation to quadratic functions plays an important role in understanding this branch of mathematics.

Provision in the Mathematics 20 curriculum for elective units was important to this study because it allowed for the inclusion of the computer literacy unit while remaining within the bounds of the curriculum. At the same time, the unit did not interfere with the overall timing of the course.

In regard to the difficulty of the unit, most of the problems may arise because students focus too heavily on the calculations involved and fail to comprehend the overriding mathematical concepts. In this study, as in other studies completed previously, programming was seen as requiring a "complete and careful analysis of the processes of the problem" (Hatfield & Kieren, 1972, p. 100). This is often more intensive than what might be undertaken otherwise. Programming should allow students to focus their attention on the central concepts of the unit while the microcomputer and computer program are used to perform the necessary calculations.

Machine and Language Selection

There are numerous machines and languages on the market, and an overwhelming amount of material often expresses conflicting views on the superiority of each machine or language. One of the most difficult decisions teachers, schools, and school systems are required to make is to commit themselves to a particular machine or language. The decision to conduct this study on the Apple IIe was based on the knowledge that these microcomputers enjoy the most widespread use in Alberta. The BASIC language was chosen for two reasons. First, it is the language housed in most microcomputers; second, it is relatively easy to learn.

In late 1981, the Minister of Education announced that the Department of Education would be supporting the adoption of the Apple microcomputer for school use. The announcement met with widespread criticism; however, since that time the number of microcomputers in Alberta has increased by 1245 percent, and Apple microcomputers are by far the most popular. Table 2 contains statistics on the microcomputers in use in Alberta schools. As is indicated by the data, Apple computers accounted for over half the microcomputers which were in use in the province in 1983. For this reason the Apple IIe was selected for this particular research project.

Table 2

Microcomputers in Use in Alberta in 1983

Machine	Number	% of Total
Apple	2077	58.8
Commodore	974	27.6
Radio Shack	181	5.1
"Others"	303	8.5
Total	3535	100.0

Source: Alberta Education, 1983, p.18.

The BASIC programming language, developed at Dartmouth College in the mid 1960's, has enjoyed widespread use on many computer systems. Today, BASIC is by "far the most widely known language for personal and small business computers in the world" (Bloch, 1981, p. 15) because the language is built into most microcomputers. The widespread use of the BASIC programming language has allowed it to withstand a large amount of criticism and, at the same time, enabled it to become the "most widely known language in secondary schools" (Spencer, 1978, p. 152). In addition, another contributing factor to its popularity at the secondary level is its simplicity: "A beginner who knows nothing about programming or computers [is able to] master the rudiments of BASIC in a few hours" (Spencer, 1978, p. 153-154).

Both the widespread use and the simplicity of the language were important factors leading to the selection of this particular language for the study. The general use and easy accessibility of the language ensured that the unit could be adapted to other microcomputers and other classroom situations, while the simplicity of the language increased the probability that it could be learned quickly. This last point was extremely important since a limited amount of time was available for learning a computer language.

Design of Materials

While the integration of computers into the educational system has been aided by the fact that it began as a "grassroots" movement, teachers generally are still poorly equipped with materials and aids to assist them with the integration of computers into various subject areas. In addition, the process has been further complicated by the widespread debate as to the exact approach that should be taken. Should computers be used to supplement the curriculum or should computer studies be carried on independently of the regular curriculum?

The perspective from which this project was conducted was consistent with the view that:

> If computers are to be fully utilized, the possibilities must arise from the context of a total curriculum rather than from isolated appendages to whatever else happens to be in use by the teacher at the time. (Williamson, 1978, p.150)

To assist in the integration of computers in this manner, Williamson has promoted the development of an "unfinished" curriculum which possesses a conceptual design backed by scholarship and a creditable theory of learning but is not bound by rigid scope and sequence. More specifically, Williamson states that the conceptual design of an "unfinished" curriculum should be "clear enough to enable teachers and scholars to adapt, modify, and go beyond the original design ... in accordance with their own talents and teaching situations" (Williamson, 1978, p. 150). His suggestions for an "unfinished" curriculum were incorporated into the design of the materials used in this research project. Specifically, most activities were designed to be used independently of one another, thereby leaving teachers with the opportunity to use those activities that were best suited to their particular teaching styles. In addition, most activities involved a discovery approach to mathematics since such an approach is said to place "much emphasis upon meaning and understanding" (Petty, 1955, p. 121).

In order to develop a suitable unit, the textbooks prescribed by Alberta Education for use in Mathematics 20 (Math Is/5 and Holt Mathematics 5) were reviewed and five subtopics, suitable for computer adaptation, were selected. The five subtopics were: investigating the shape of a function, graphing, completing the square, the roots of functions, and using the discriminant. Each of these sub-topics consisted of one or more activities adapted from the prescribed textbooks. In the first activity on the shape of a quadratic function, the students were required to write a program in BASIC and then to evaluate a number of functions using the program. The three activities on graphs of functions provided for graphing different forms of the function using Lund and Andersen's (1982) Computer Graphing Experiments program. The single activity on completing the square was designed as a student follow-up to a teacher-centered activity. Three separate activities on roots focused on searching the interval within which roots lie and applying the quadratic formula. Finally, the fifth sub-topic consisted of the use of the discriminant to examine the nature of the roots of the function.

The instructional unit consisted of activities for students and a guide for the teacher. A complete set of materials is included in Appendix A.

B. TESTING THE MATERIALS

In order to address the major research questions, an appropriate site had to be selected for testing the materials. Furthemore, specific procedures had to be developed which included defining the role of the researcher.

Site Selection

Permission was obtained from the Edmonton Separate School Board to conduct the study in one of the high schools. Through contact with a particular principal and teacher, arrangements were made for testing the materials with a class of twenty-four Honors Mathematics 20 students. Since this was an Honors class the likelihood of all students mastering the BASIC language was increased, and the incidence of proactive inhibition during the second phase of the study was decreased. However, since the class likely would have been highly successful under regular instructional procedures some resistance to programming as a learning activity was anticipated. Of the twenty-four students, only fifteen students who attended all computer sessions and wrote both the pretest and the posttest were included in the analysis.

Procedures

To maximize the use of computers in the high school in which this study was conducted, a computer laboratory housing twenty Apple IIe computers and one printer had been established. Since the study required the instructional unit to be supplemented with regular classroom instruction, it was necessary for the class to move between the classroom and the computer laboratory. In order to minimize the number of moves, the teacher and researcher agreed to alternate lessons between the two rooms. Furthermore, the teacher agreed to conduct the classroom lessons while the researcher agreed to assume responsibility for all sessions conducted in the computer laboratory. Therefore, the research project was conducted as an observational case study in which the researcher also served as the teacher for the specific unit under study. All observational notes were recorded immediately following each teaching session. During the materials testing sessions, which were conducted in the computer laboratory, the classroom teacher was also present and participated actively in assisting students when they encountered difficulty. The teacher did not have any preparation in computer programming and regarded the project as a learning opportunity.

Two units were involved: the computer literacy unit, and the unit covering quadratic functions. The computer literacy unit, which was integrated into the course through the elective component available in Mathematics 20, was concerned with familiarizing students in the use of the microcomputer and teaching students programming in BASIC. The time duration of this unit was approximately 150 minutes (three days of 50 minute classes). This unit played an important part in the success of the remainder of the study because it afforded the researcher with time to develop a good working relationship with the teacher involved, time for the students to become comfortable with the presence of the researcher in the classroom, and time for the researcher to become familiar with the students.

The second component, the quadratic functions unit, comprised the main part of the study. During this unit students were required to complete the majority of their assignments on the microcomputer either through the use of an interactive program or by writing and running their own computer programs. This portion of the study required 950 minutes (nineteen 50 minute classes). Approximately half the time (500 minutes) was spent in the computer laboratory.

Test Validation

Student achievement was measured by the administration of a pre-test and post-test to the students involved in the study. In addition, both tests were administered to another group of Mathematics 20 students to assess the validity of the questions. Since numerous differences existed between the two groups of students, only the results for the students involved in testing the materials are reported in Chapter Four.

IV. RESULTS OF THE CLASSROOM TEST

The classroom test of materials involved two units of the Mathematics 20 course: the computer literacy unit and the unit on quadratic functions. Following a brief description of the literacy unit, the three major research questions are addressed in this chapter. The extent to which the computer programming approach promoted relational understanding is discussed for each of the five sub-topics. An analysis of test results is also presented. Reactions of both the students and the teacher to the use of computer programming are then described. In general, the chapter addresses the three research questions which guided the study.

A. COMPUTER LITERACY

Although the questions posed for the test did not focus on computer literacy, the unit did play an important role in preparing students to complete the computer assignments in the quadratic functions unit. Because of the fundamental importance of this unit, the results warrant some elaboration.

As previously mentioned, three class sessions (150 minutes) were allotted for learning the fundamentals of programming in BASIC. For those students who already possessed a good working knowledge of computers and computer programming, the three days were more than sufficient; however, for those students who were not familiar with computers or computer programming, the time allotted was insufficient. Specifically, this second group of students required more time on the computers to increase their confidence and to consolidate their knowledge. By the end of the programming sessions, three groups of students could be identified. The first group, approximately one third of the class, was very familiar with computer programming and exuded a high degree of confidence. The second group of students, approximately half the class, expressed interest and willingness to learn; however, confidence was lacking. The remaining students expressed neither interest nor desire to learn programming and were absent for two or three of the initial days on the computers. Whether the limited interest was the result of missing these initial sessions, or if missing the sessions was the result of their lack of interest, is unknown.

B. COMPUTER PROGRAMMING AND UNDERSTANDING

A general intent of the classroom test was to assess the usefulness of the materials which were developed. From a more conceptual and methodological perspective, the test focused on the following question: Does the use of the computer and the integration of computer programming into the curriculum promote a relational understanding of mathematics? The question was addressed through the observation and assessment of the ways in which students performed on the activities included in the instructional materials.

The activities which students completed (see Appendix A) centered on five concepts covered in the quadratic functions unit. Since each concept emphasized different course content and, therefore, different understandings, each concept is reviewed separately. The review of each topic consists of a brief summary of classroom observations and a discussion of the understanding(s) which were stressed. The discussion of understanding is based on the theories of Greeno(1978) and Skemp(1971) which were elaborated in Chapter Two.

Investigating the Shape of a Quadratic

Due to time constraints, less than one class period or approximately 25 minutes was spent on this particular topic. The computer program (Activity One, Question One) was written as a class assignment to increase the confidence of those students experiencing difficulty with writing computer programs and to clarify the researcher's expectations. Although the students had discussed the concepts of maximum, minimum and vertex, they were slow to apply their knowledge to the data generated during this activity. Activity Two was not completed due to limitations on available time.

The primary purpose of these activities was to introduce students to the general shape of a quadratic function and to the concepts of maxima, minima, and vertex. Since the students had already discussed these concepts in class, the purpose of the lesson was redefined as follows: To have the students apply their knowledge of maxima, minima and vertex, and to demonstrate to the students that the experiences occurring in the computer laboratory were extensions of the experiences that were occurring in the regular classroom. The results of these activities showed that for the students the experiences in the regular classroom and those in the computer laboratory lacked coherence. The students had difficulty relating their knowledge of maxima, minima, and vertex to the material under discussion in the laboratory setting.

Graphing Quadratic Functions

The purpose of the activities on graphing was for students to determine the relationship between the variables a, p, and q in the equation $F(x) = a(x - p)^2 + q$ and the location of the graph represented by the equation. Two class sessions (total of 100 minutes) were devoted to these activities.

Many students did not see a need to discover the relationship between the equation and the graph. This may have been partly because the material had been covered in class prior to the computer session and partly because the students were not accustomed to being placed in the position of discoverers of knowledge and of relationships. However, while some students seemed uninterested in the visual experience provided by these activities, others were extremely interested in exploring the various graphing options available on the programs. Specifically, these students appreciated the opportunity to explore this area of mathematics without the tedium of plotting the graphs for themselves.

The main distinction between the observations made during this activity and those made during the previous one is that here students quickly related the investigations to material covered previously in class. Coherence between the experiences in the classroom and the computer laboratory was much greater for the students during this activity. In addition, Greeno might argue that those students who explored other types of graphs were making important mathematical connections.

Completing the Square

Since the first major progamming assignment was included in this section of the unit, two class periods (total of 100 minutes) were allotted for its completion. During the first period, students were required to write, revise, and test their programs. On the second day, the class reviewed the procedure by contributing to the writing of a complete "finished" program. The students then completed the remaining questions by using their own programs or the "class" program. A two-day approach to writing programs and completing assignments was used for the duration of the project. This particular approach was found to be very successful because it allowed the students time to develop their own ideas and also gave them the opportunity to correct any misconceptions prior to running their programs to solve problems.

The process of writing a program proved to be engrossing and satisfying for the students, especially when a successful program was the result. For most students, however, the process also resulted in an encounter with mathematics. Three common misconceptions were clarified during this assignment.

1. The axis of symmetry of a quadratic function is represented by -B/2A. For many students, -B was understood as meaning "the opposite of"; therefore, if B is 4, then -B is -4 since -4 is the opposite of 4. When programming, however, this definition was not functional, and students had to re-think their definition to determine the operation involved.

2. A second common mistake with the same formula for the axis of symmetry (-B/2A), involved specification of order of operation. Verbally most students realized the equation requires the numerator to be divided by 2*A; however, in programming, often the order of operation was not specified thus yielding an incorrect answer.

3. In addition to specifying the vertices, axis of symmetry, and maximum or minimum value, the students were

required to specify whether the vertex represented a maximum or a minimum. When writing the program for this activity, students either failed to check whether the value would be a maximum or minimum and stated the vertex was always a maximum or continually based their decision on the y coordinate at the vertex, i.e., if the y value at the vertex was negative, the vertex was a minumum; if the y value was positive, the vertex was a maximum.

Some common errors and misconceptions are illustrated by the following two sample programs:

Program No. 1

```
10 INPUT A,B,C
20 ? "X = "; -1*B/(2*A)
30 ? "X="; -1*B/(2*A)","(4*A*C) - B*B/(4*A)
40 ? "MAX = "; (4*A*C) - B*B/(4*A)
50 END
```

Program No. 2

```
10 INPUT A,B,C
20 PRINT "AXIS OF SYMMETRY: X ="; -1*B/2*A
30 LET J = -1*B/2*A
40 LET R = 4*A*C - B*B/4*A
50 PRINT "VERTEX: ("J","R")"
60 IF R < 0 THEN PRINT "MAXIMUM";R
70 IF R > 0 THEN PRINT "MINIMUM";E
80 END
```

In both sample programs -B is defined as the operation -1*B; however, in the first program the vertex is always a maximum (Line 40). In the second program the order of operation needs to be specified (Lines 30 and 40), and the method of determining if the vertex is a maximum or a minimum is incorrect (Lines 60 and 70).

The following program illustrates a well-developed understanding of the concepts to be included in the program and the nature of their relationships:

```
5 HOME
10 ?"INPUT THE VALUES FOR A,B, and C FOR:"
15 ?"AX*2 + BX + C"
20 INPUT A,B,C
25 IF A = 0 THEN ?"QUADRATIC PLEASE": GOTO 10
30 ?"AXIS OF SYMMETRY: X ="; -1*B/(2*A)
40 ?"VERTEX:(";-1*B/(2*A);",";(4*A*C - B*B)/
         (4*A);")"
49 ?"RANGE"
50 IF A > 0 THEN ?"Y >= ";(4*A*C - B*B)/(4*A)
60 IF A < 0 THEN ?"Y <= ";(4*A*C - B*B)/(4*A)
65 ?"HIT RETURN TO CONTINUE"
67 INPUT R$
70 GOTO 5
```

This sample program illustrates a good understanding of the concepts involved. In Line 25 the student first checks to make certain that the function is a quadratic. Lines 30 and 40 correctly determine the axis of symmetry and vertex, making certain to specify the order of operation. The maximum or minimum value is calculated and stated in Lines 50 and 60.

The activity of writing a computer program proved to be a true test of the students' understanding of the material that had been covered to that place in the unit. The coherence and correspondence of the students' conceptions were tested by the writing of programs, especially in relation to determining maximum and minimum points. Although the students had been provided with many visual experiences, they were unable to conceptualize a graph with a maximum point below the x axis and a minimum point above the x axis.

In addition to coherence and correspondence, the activity of writing a computer program also challenged students to make connections between this area of algebra and those studied in earlier years such as order of operation and additive inverse. Students' connections were also challenged during this activity by the generality required in their computer programs. Generality, according to Greeno, is one of the three distinguishable ways through which we can assess the understanding of procedures.

If these same observations are related to Skemp's distinction between instrumental and relational understanding, they seem to support the theory that relational understanding is promoted by the writing of computer programs. To complete the assignment students were required to clarify for themselves and to specify to the computer the exact calculations involved in the equation for the axis of symmetry (x = -B/2A). The students quickly discovered that an instrumental understanding (-B means the opposite of B) did not suffice and, as a result, they were required to rethink their definitions and to understand the mathematics involved in this equation.

The students' misunderstanding of maximum and minimum points was also clarified through this activity. As was indicated earlier, many students incorrectly assumed that the y coordinate of the vertex determined whether the parabola possessed a maximum or minimum point, i.e., a positive y coordinate indicated a minimum point, and a negative y coordinate indicated a maximum point. By observing the students' programs, the teacher and the researcher were able to identify this misconception and to clarify for the students the relationship between the equation of a parabola and whether it possesses a maxiumum or a minimum point. Unfortunately, the students were unable to clarify this misconception by themselves; however, the teacher did comment that the exact nature of the students' misunderstanding was easy to identify through an examination of their computer programs.

Roots of Quadratic Functions

Three distinctly different approaches to determining the roots of a quadratic function are contained in this section of the unit. Two of the approaches, determining roots by searching intervals and the quadratic formula, were used in this research project.

The method of searching intervals to determine roots proved to be difficult for those students who failed to understand an important concept. Specifically, if the range value of a function changes from negative to positive or positive to negative in an interval, then the graph must cross the x axis in that interval. For students who understood this concept, however, the activity proved to be quite successful. During the second class period which was devoted to this activity, an excellent discussion developed around the following question: If 4 and -4 are roots of a quadratic function, how do we know the axis of symmetry is 0? For many students, the discussion clarified and extended their understanding of symmetry in regard to quadratic functions. The activity concluded by having students adapt their programs to determine the roots of cubic and quartic functions. Most students had little difficulty in extending their programs.

The second major programming assignment required students to write a program which determined the roots of a quadratic function by using the quadratic formula. In general, students showed more confidence and ability in programming when completing this assignment. Again, however, problems arose in regard to their understanding of mathematics; several students had difficulty determining how the symbol ± could be included in a computer program, and most students failed to check for the occurrence of a negative discriminant. Consequently, they had difficulty running their programs with some of the test data. For those students who had difficulty interpreting the symbol ±, the problem was resolved when they began to view the symbol as being representative of a series of operations. Students who experienced difficulty when a negative discriminant was encountered quickly recognized the need to check for such an occurrence when the resulting error message was understood. In addition, during the programming assignment, two misconceptions were clarified. Specifically, the misconceptions were:

1. if A is negative, then the polynomial is not a quadratic function; and,
2. if the value of the discriminant is zero, then the quadratic function has no real roots.

The program listed below illustrates these misconceptions:

```
10 INPUT A,B,C
15 IF B*B - 4*A*C = 0 THEN 65
17 IF A < 0 THEN 60
20 X = (-1*B + SQR(B*B - 4*A*C))/(2*A)
25 Y = (-1*B - SQR(B*B - 4*A*C))/(2*A)
30 PRINT "ROOTS ARE: "; X,Y
55 END
60 PRINT "NO ROOTS"
65 PRINT "NOT A QUADRATIC"
```

The program contains incorrect methods for determining if a polynomial is a quadratic function (Line 15) and if the quadratic function has real roots (Line 17).

In the concluding activity of this section of the unit, students were required to determine the relationship that exists between the coefficients of a quadratic function and the sum and the product of the roots. Since this topic had not been covered in class, the activity engaged students in a real discovery situation. The majority of students quickly discovered the relationship. In doing so, they displayed considerable growth in confidence and ability both in regard to their use of computers and as discoverers of knowledge. While completing the activity most students, for the first time, began to use the computer to verify answers when they were not specifically required to do so.

In designing the activities for this concept, three methods for determining roots of a quadratic function were included in order to increase students' understanding of the roots of a function. The two initial activities were also intended to provide students with a better understanding of what the answers arrived at by the quadratic formula actually represent. In retrospect, this series of activities provided students with the optimum opportunity to increase coherence, correspondence, and connection.

Internal coherence of the material was increased for the students by completing two different activities on determining the roots of a quadratic function. In addition, discussions on the relationship between the roots of a quadratic and the vertex of the parabola strengthened students' understanding of the concept of symmetry. Again, as in previous activities, correspondence could be verified by the correctness of the algorithm students were using in their computer programs. Understanding as a connection was displayed by students in two different ways during the course of these activities. First, as was explained earlier, connection was shown through the generalizations students made in their computer programs. Second, it was displayed when students quickly adapted their programs in the first activity to determine the roots of cubic and quartic functions. Their understanding of roots was further demonstrated when they knew a cubic function would have at most three roots whereas a quartic function would have at most four roots.

The Nature of the Roots

Only a few students had the opportunity to attempt the final activity on the nature of the roots. Most of those who did were successful in drawing conclusions. The major difficulty students experienced in this activity was visualizing the graph of a quadratic function which possesses identical roots. In this regard, the graphing exercises which concluded the activity proved to be very helpful.

For students who had the opportunity to complete this particular activity, the benefit seemed to lie in increased coherence between a function, its graph and its roots. Specifically, students tended to realize the information a graph could provide regarding the roots of a function and the information the roots of a function could provide concerning the location of the graph.

C. STUDENT ACHIEVEMENT

In addition to observing the performance of students in class on the various programming exercises, a topic by topic achievement analysis was also carried out. Both a pre-test and a post-test on quadratic functions were administered to students involved in the study. The tests, which were not parallel forms, were initially validated on another class of Mathematics 20 students. Summaries of responses to questions on both tests are presented in Appendix B.

Pre-test Results

The results of the pre-test summarized in Table 3 indicate that prior to beginning the unit, the students involved in the study already possessed a good working knowledge of graphs and their equations. This was evident in the overall results of the test: out of a total of 165 responses, 132 or 80 percent were correct.

On the last question of the test which required the students to match the graph with the correct equation, fourteen of the fifteen respondents were able to complete the matching successfully either by substitution of values or completion of a table of values. At the time, only one student indicated that the coefficient of the second degree term could be used to determine whether the graph opened up or down.

The only question on the pre-test which related to the characteristics of quadratic functions was Question 2. Specifically, this question required students to analyze a series of coordinates and to determine whether or not they lay on the graph of a specific function. Included in the data were two coordinates which did not coincide with the graph. Twelve of fifteen students were able to identify the point (-4, 14.3) as an error since the function $f(x) = 3x^2 + 9x + 2$ must be statisfied by an integral value when $x = -4$.

Table 3

Summary of Pre-test Results

Question	Correct/Incorrect Responses	Comment
1.	14/1	All students substituted to determine the value of the function.
2. a)	12/3	Twelve students recognized 14.3 as an error. Ten of the twelve stated that all values must be integers.
b)	7/8	Seven students recognized 36 as an error: five substituted to determine the correct value, and two students stated that the values must be symmetric about 0.
3.	14/1	Thirteen students substituted to determine the correct value, and one student used the quadratic formula.
4. a)	14/1	Only one student identified both +12 and -12, thirteen identified +12, and one student factored the trinomial to (x+9)(x+4).
b)	12/3	Twelve students correctly identified 12 as the missing term, one student indicated the missing term was 3, two students attempted to factor the trinomial.

Table 3 (continued)

5.	12/3	Of the correct responses, eight students substituted values, three students factored the equation, and one student used the quadratic formula.
6. a)	12/3	Ten students factored the quadratic, and two students solved the equation.
b)	8/7	Eight students factored the quadratic, and one student solved the equation.
c)	13/2	Twelve students factored the quadratic, and one student used the quadratic equation.
7. a)	14/1	All students completed the first four parts of the question correctly. One student did not complete the last two parts.
b)	--/-	Nine students indicated that they substituted values, one student applied minimum and maximum values.
Total	132/33	Eighty percent of the responses were correct.

However, only 47 percent of the students (7 of 15) correctly identified the second error in the data since recognition required students to be aware of the symmetric nature of quadratic functions. Furthermore, of the seven students who were able to recognize the error, only two stated that they had applied this principle. The other five successful students, contrary to the instructions, had checked each point by substitution of values.

There were no questions in the pre-test concerning the zero values of a function since this concept had not been covered in any previous mathematics course. The concept is first introduced in the unit on which the study was based.

Post-test Results

During the course of the study, a post-test (see Appendix B) was developed which took into account the fact that the students possessed a good initial understanding of the graphs of quadratic functions. Consequently, the questions on the post-test differed from those on the pre-test. The specific purpose of the post-test was to determine whether or not the students understood the role of the coefficients in determining the placement of a graph on the Cartesian plane. In addition, several questions were included which focused on either the symmetric nature of quadratic functions or the roots of quadratic functions. The results of the post-test are summarized in Table 4.

Table 4

Summary of Post-test Results

Question	Correct/Incorrect Responses	Comment
1. a)	14/1	One student sketched a graph with the vertex in Quadrant 1.
b)	15/0	All students indicated that the graph never touched or crossed the x axis.
2. a)	11/4	Eleven students indicated "a" was negative, two indicated incorrect values for "a" and "c", one stated an incorrect equation, and one student did not respond.
b)	13/2	Thirteen students sketched the graph correctly; two students did not complete the question.
3.	11/4	Nine students indicated that the values were not symmetric; two students indicated that the point (-1,1) was lower than the minimum. Of the four incorrect responses, two students did not respond to the question.
4.	9/6	Six students indicated that the points were symmetric, and three indicated that the points were below the maximum. Three students did not respond.

Table 4 (continued)

5. i)	10/5	Five students stated the correct values of "a" and "k" but an incorrect value for "h".
ii)	9/6	Five students stated the correct values of "a" and "k" but an incorrect value for "h". One student indicated incorrect values of "k" and "h" but the correct value for "a".
iii)	10/5	Four students indicated the correct values of "a" and "k" but an incorrect value for "h". One student stated the correct value of "a" but incorrect values for "k" and "h".
iv)	11/4	Four students indicated the correct value of "a" but incorrect values of "k" and "h".
v)	10/5	Three students indicated the correct values of "a" and "k" and the incorrect value of "h". Two students stated the correct value of "a" but incorrect values for "k" and "h".
6.	2/13	Two students were successful in completing the question. Five used the correct method but did not complete the question.
7.	15/0	Ten students indicated that a quadratic function cannot have three roots. Five students stated the values were not symmetric.

Table 4 (continued)

8.	11/4	Eleven students stated the values were not symmetric.
9.	9/6	Nine students indicated the values do not cross the x axis. Two indicated that the graph does cross the x axis, one indicated that the function has no roots, and three did not respond.
10	22/1	Eight students explained that roots of an equation occur where the graph cuts the x axis or where y = 0. One indicated that a cubic function has three roots, and three did not respond.
11.	10/5	The students used a variety of approaches: substitution (5 students); quadratic formula (1); axis of symmetry (7); working back from the roots (1). One student did not respond.
12.	14/1	Eleven students substituted to determine the answer, one used the quadratic formula, one student indicated that 9 must be positive, and one student did not provide an explanation. One student did not respond.

Table 4 (continued)

13. a)	13/2	Fourteen students indicated that the vertex is a maximum: thirteen indicated that the graph had one x intercept, one student indicated there were two intercepts. One student did not respond.
b)	10/5	Fourteen students indicated that the vertex was a maximum: ten indicated that there were two intercepts, one indicated that there was one intercept and one indicated that there were no intercepts. Three students did not respond.
c)	8/7	Fourteen students indicated that the vertex was a minimum. Eight indicated that there was one intercept, four indicated that there were two intercepts, and three did not respond.
d)	9/6	Nine students indicated that the vertex was a maximum and that the graph intercepted the x axis in two points. Five students did not respond.
Total	246/92	Seventy-three percent of responses were correct

Three questions in the post-test (Questions 1, 2, and 5) were designed to check the students' understanding of the graphs of quadratic functions. The first two questions focused on the ability to analyze the graphs of functions of the form $f(x) = ax^2 + c$ while the last question related to graphs of the form $f(x) = a(x - h)^2 + k$. Responses to these questions indicated that students understood better the role of the coefficients in determining the placement of functions of the first form (13 of 15 students) than of the second (10 of 15 students, on average).

With respect to the symmetric nature of quadratic functions, the results of the post-test indicated that students were more likely to look for this characteristics at the completion of the unit than they were at the outset. Observation of this characteristic formed the basis of Questions 3, 4, and 5. Analysis of the questions on the roots of quadratic functions (Questions 7, 9, and 10) indicated that all fifteen students understood that quadratic functions possess two roots and that the roots occur at the point where $y = 0$ or where the the graph crosses the x axis. Some students (8 of 15) were able to apply their knowledge of the roots of quadratic functions to Question 10 which involves the roots of a cubic function.

Although there was some variation in performance, most students demonstrated increased ability to interpret graphs at a more abstract level and to apply their knowledge of the characteristics of quadratic functions on the post-test.

Given that the students already possessed a good knowledge of graphs at the beginning of the unit, the improvement was not dramatic. In retrospect, this could have been anticipated. A similar ceiling effect was observed by Kieren (1969) with high achievers in an earlier study.

D. STUDENT REACTIONS

The second research question addressed in this study focused on student reactions to integrating computer programming into the mathematics curriculum. The reactions were solicited at the end of the project through the administration of the student questionnaire contained in Appendix C. The specific purpose of the questionnaire was to obtain feedback from the students in regard to the project, the use of computers in mathematics, and their general attitude toward computers and their role in society. All of the students who responded to the questionnaire stated that they had previous experience with computers. Eleven students reported that they had completed a computing course, and five reported they had access to a personal computer at home. The results of the questionnaire are contained in Table 5.

Table 5

Summary of Responses to the Student Questionnaire

	SD	D	N	A	SA
a) I enjoyed using the computer.	0	0	0	8	6
b) I feel confident about my ability to use computers.	1	1	0	9	3
c) The time I spent on the computer was worthwhile.	1	1	3	7	2
d) Writing computer programs helped me learn.	1	1	1	8	3
e) Using the computer helped me learn more about math.	0	5	2	6	1
f) I would like to use the computer more in math.	0	5	2	5	2
g) I prefer to work by myself on the computer.	1	5	3	3	2
h) I enjoyed writing computer programs more than I enjoyed using the prepared programs like the graphing programs.	0	4	2	3	5
i) There was adequate time to complete the assignments.	0	3	0	9	2
j) Having two people teach the unit was confusing.	4	2	6	2	0
k) The regular teacher should teach both the classroom and microcomputer sessions.	2	1	6	4	1

Table 5 (continued)

l) Every student should have some minimal understanding of computers.	0	1	0	6	7
m) Every secondary school student should be able to write a simple computer program.	0	2	1	6	5
n) Computers can be useful instructional aids in many subject areas other than mathematics.	0	0	1	8	5
o) The use of computers in education results in less personal treatment of students.	1	4	4	5	0
p) Computer are gaining too much control over people's lives.	4	4	2	3	1

As is illustrated in the table, all of the students stated that they enjoyed using the computer and, in general, felt that the experience was worthwhile and educational. The students were divided as to which particular activity was most interesting and beneficial. Several students stated that they enjoyed the visual experience provided by the graphing activities while others favored the programming activities. The students who favored the programming activities commented that they found the process of writing computer programs extremely challenging and found many situations arising in which they could learn from their mistakes. Several students stated that their understanding of mathematics improved through the process of writing a computer program while others appreciated the speed and precision with which results were obtained. When asked to comment on problems or difficulties which arose during the course of the research project, two students expressed a lack of confidence in their ability to use computers and, in particular, indicated that they found writing and understanding programs extremely difficult. Several other students commented negatively on the explicitness and detail required in a computer program.

The students' responses to questions concerning the use of computers in mathematics were generally divided. No consistent indication, either negative or positive, was evident with regard to the extended use of computers in mathematics or the approach that should be taken if computers were used. Specifically, two questions were left unanswered:

1. Should computer implementation use a team approach to teaching or should the classroom teacher be responsible for both the classroom and the computer sessions (Questions j and k)?
2. Should students be encouraged to work independently at the computers, or is a group approach to computer programming more effective (Question g)?

The preferences of students in relation to each of these questions were varied.

The students' opinions were more definite with respect to general attitudes toward computers. Specifically, the students agreed that every secondary student should possess a minimal understanding of computers and should be able to write computer programs. The use of computers in subject areas other than mathematics was strongly supported by the students. However, responses did not reflect consensus on issues such as whether computer use resulted in less personal treatment or whether computers were gaining too much control over people's lives.

E. TEACHER REACTIONS

The third purpose of the classroom test was to determine the teacher's reactions to the integration of computer programming and computer laboratory experiences into the mathematics curriculum. The teacher's reactions were solicited during the course of the study through conversations following the computer sessions and at the completion of the study in an interview. The discussions centered on the following topics:

1. the value of the graphing activities;
2. the identification of student errors; and,
3. the use of a team approach versus the teacher assuming responsibility for both the classroom and computer laboratory session.

The points raised through these discussions warrant brief elaboration.

During the study only a limited amount of time was devoted to the graphing activities since the topic had already been covered in class. Consequently, the teacher felt it would be more beneficial to proceed immediately to the programming activities. At the end of the study the teacher commented that the graphing activities were probably the most worthwhile. Specifically, the teacher noted that using the computer to sketch graphs allowed students to concentrate on the concepts being studied rather than on the mechanics of graphing. This, in turn, resulted in developing a higher degree of knowledge about quadratic functions and their graphs. In addition, students had more time to pursue related topics.

In regard to the identification of student errors, the teacher commented favorably on the ease with which errors and misconceptions could be identified in student programs.

The teacher noted that in correcting student assignments what is often brought to a student's attention are specific mistakes such as calculation errors rather than the diagnosis of general errors in procedure. Correcting a student's computer programs, however, required the instructor to identify and to diagnose general procedural errors.

The last major topic of discussion concerned whether the team approach (teacher and computer specialist) should be used to integrate computer programming into the mathematics curriculum or whether the teacher should assume responsibility for both the classroom and computer sessions. It was the feeling of the teacher involved in this study that the two experiences would be intergrated much more effectively if the teacher were to assume full responsibility. At the time of the study, however, this teacher did not feel confident to assume the responsibility without assistance, especially in the form of appropriate classroom materials. In future, he intended to incorporate computer assisted graphing activities into the unit on quadratic functions.

In general, the teacher commented that he looked forward to new developments in the area of computers and mathematics. As a result of the study, he became more conscious that students could experience mathematics in greater depth and variety than occurs through more standard forms of mathematics instruction.

V. SUMMARY AND CONCLUSIONS

The first section of this chapter presents a summary of the general approach taken to developing and testing a set of materials designed to integrate computers into a mathematics unit. A number of conclusions about the ways in which the activities contributed to increased understanding are presented in the second section. The third section outlines several possible areas for further research and additional developmental activities suggested by the present study.

A. PROJECT SUMMARY

In 1980 the National Council of Teachers of Mathematics published An Agenda For Action which stressed that computers should receive increased emphasis in mathematics education. While the availability of computer-related mathematics materials for use in the classroom is increasing, these materials are often found to be unsuitable for a variety of reasons. The purpose of this research project was to develop and test a Mathematics 20 unit on quadratic functions in which computer activities were integrated wherever they were deemed appropriate. The unit was developed in accordance with the Mathematics 20 curriculum as prescribed by the Alberta Department of Education.

The materials developed for this study were focused on five topics studied in quadratic functions. These topics were as follows: the shape of the graph of a quadratic function, graphing quadratic functions, completing the square, the roots of quadratic functions, and the nature of the roots. Built into the design of these materials was the intent to have the students access the computer in two different ways: first, through the use of a commercial graphing program, and second, by writing their own computer programs. The graphing activities were designed to emphasize a discovery approach to mathematics while the activities which required students to write a computer program were intended to increase their understanding of mathematics.

The instructional materials were field tested with a class of Honors Mathematics 20 students. Before the materials were actually introduced, the students involved in the study received three hours of instruction on the operation of the Apple IIe microcomputers and the BASIC computer language. Procedures for the study involved having the teacher provide regular classroom instruction and the researcher conduct the computer sessions. The computer activities required use of either a commercial program, such as Computer Graphing Experiments (Lund & Andersen, 1982), or a student generated program which used specific data as a test of accuracy and usefulness.

On the basis of the assessment, the unit appears to have been reasonably successful in terms of the amount of learning that resulted as well as in terms of the attitudes of both the teacher and the students toward the unit. In regard to student learning, the results indicated that computer integrated activities may increase student understanding. Specifically, the results tended to show that the activity of writing a computer program requires students to comprehend fully what is involved in each and every calculation which they wish to perform. This high degree of specificity required the students not only to focus on new concepts but also demanded that they fully understand concepts learned earlier. In regard to the attitudes of both the teacher and the student, the unit was viewed as a rewarding experience; however,the majority of students indicated that the classroom and computer sessions would be better integrated if the teacher assumed full responsibility for all activities.

B. CONCLUSIONS

The main research question addressed focused on whether the integration of computers into the mathematics curriculum could be used to increase students' understanding of the subject matter under discussion. As was outlined in Chapter Two, Greeno (1978) has identified three criteria which may be used to judge the degree of understanding: coherence, connectedness, and correspondence. The observations made during the course of this study illustrated that these activities benefitted the students most in the areas of coherence and connection. Coherence, in Greeno's terms, refers to the relationship between parts. For mathematics educators this is an important criterion since mathematics often appears to be taught in a void with no relationship between concepts being demonstrated. The importance of this criterion in regard to this study was that through the course of the unit students began to see that mathematics could be discussed in different terms and situations. At the beginning of the unit the students regarded the classroom sessions and computer sessions as independent activities. As a result, they were generally unsuccessful in completing the assignments. Once the students began to see the relationship between the two experiences and to realize that mathematics could be discussed in different ways, they were much more successful with their assignments. The third criterion, that of connectedness, is defined by Greeno as consisting of three different elements: connection between general mathematical concepts, connection of mathematics to ordinary experience, and connection to general schemata. The results of this study indicate that the activity of writing computer programs improved the students' connection between general schemata. In particular, the observations suggest that connections between existing schemas were either established or improved.

In regard to Skemp's (1976) distinction between instrumental and relational understanding, the observations made during the test support the claim that computer activities increase a students' relational understanding of mathematics. Specifically, the basic reason for the improved relational understanding is that the activity of writing a computer program does not permit students to rely on coping strategies to succeed. Instead, they must explain to a "difficult taskmaster" exactly what they want to do. For the students involved in this study, the process of writing computer programs resulted in increased understanding of the following concepts:

1. -B means -1*B rather than the opposite of +B;
2. when working with formulas, order of operation must be specified rather than assumed; and,
3. maximum and minimum values of quadratic functions are dependent upon whether the graph opens up or down and are not determined by the location of the vertex.

In addition, the ability to generate graphs of specific functions with the aid of the computer proved invaluable as the unit progressed since many graphs could be sketched with very little effort in a minimum amount of time. The graphs were then used to clarify student misconceptions of how the variables in the equation $f(x) = a(x - h)^2 + k$ affect the placement of the graph of the function.

In summary, the results of the study support Papert's (1981) claim that when computers are integrated into the mathematics curriculum the students are placed in a fundamentally different relationship with regard to mathematics. Furthermore, the manner in which the activities were integrated into the curriculum support the view that it now

> seems feasible to try to work toward a curriculum in which activities and materials designed to improve students' skills and those designed to improve their understanding can be integrated and made to be synergistic rather than competitive and antagonistic. (Greeno, 1978, p. 282)

Specifically, the results suggest that students' understanding of mathematics can best be increasd by changing their relationship to mathematics from being passive participants to being active explorers.

C. FURTHER RESEARCH AND DEVELOPMENT

This study provides only a limited glimpse into the area of computer application in mathematics education. The results and conclusions suggest several related areas worthy of further research. One of the limitations of this study was the short time frame and small sample size. Obviously, before any major conclusions can be drawn, research projects will have to be carried out in greater depth for a longer period of time to assess longitudinal effects. In addition, similar studies should be conducted with other students, grades and topics.

The results of this study are not generalizable beyond similar classes of Honors Mathematics 20 students. Further studies could explore the effects of such a unit on other groups of students such as those taking Mathematics 23. In addition to the questions posed in this study, future studies might also focus on the value of the graphing activities to determine if they are more beneficial to some students than to others.

The project has demonstrated the feasibility of developing computer-oriented instructional materials which consist of a combination of those available commercially and those prepared by the teacher or developed by the students as they are learning. Many more projects of this type could be initiated and could be based, in part, on the materials which were developed for this project. Future studies should strive to delve deeper into ways in which computers may be applied in teaching mathematics and ways in which understanding of mathematics may be enhanced.

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APPENDIX A

Instructional Unit on Quadratic Functions

LEARNING ABOUT QUADRATIC FUNCTIONS
THROUGH COMPUTER PROGRAMMING

Karen L. Miklos

An Instructional Unit for Mathematics 20

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INVESTIGATING THE SHAPE OF A QUADRATIC FUNCTION

The activities on evaluating quadratic functions are intended to focus students' attention on the shape of a quadratic function through the observation of points contained on the graph of each function. The students should be able to make predictions about the shape of the graph of a quadratic function and should arrive at the following conclusions:

a. quadratic functions have a maximum or minimum point; and,
b. a quadratic function is symmetric about the maximum or minimum point.

Following the activity students should be formally introduced to the terms vertex, axis of symmetry, maximum point, and minimum point.

Suggested Approach

The first question of the activity can be assigned to students to work on individually or completed as a class activity depending upon the programming background of the students. The second and third questions can then be assigned to the students to complete on an individual basis. After the students have had sufficient time to complete Questions Two and Three, use their responses to Question Three to begin a class discussion on quadratic functions. Students will have hopefully observed from their data that quadratic functions possess a maximum or minimum point and symmetry about this point. These observations can be used to introduce the terms vertex, axis of symmetry, maximum point, and minimum point.

The discussion of the shape of quadratic functions could be enhanced by the use of a computer graphing program to graph the functions students have been investigating. The graphs would verify their conclusions and provide a visual link between the activities included in this section and the activities that follow. Activity Two has been included for this purpose.

Suggested Software

Lund, C. & Andersen, E.D. **Computer graphing experiments** (Vol. 1 and 3). Don Mills, Ontario: Addison-Wesley, 1982.)

ACTIVITY ONE

INVESTIGATING THE SHAPE OF A QUADRATIC FUNCTION

1. Write a BASIC program to evaluate a quadratic function in the domain -5 < = x < = 5 and print out both the domain and range values. Begin your program with the statement: 10 INPUT A,B,C.

2. Each of the following equations defines a quadratic function. Use your program to evaluate these functions and record the results.

a) $f(x) = x^2$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b) $f(x) = x^2 + 3x + 3$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

c) $f(x) = 2x^2 - 4x + 4$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

d) $f(x) = 3x^2 + 6x + 6$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

e) $f(x) = -x^2 - 2x - 2$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

f) $f(x) = -2x^2 + 8x - 9$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

g) $f(x) = -3x^2 - 6x - 1$

x	f(x)
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

3. By looking at the range values that have been computed, what do you notice about these functions?

ACTIVITY TWO

INVESTIGATING THE SHAPE OF A QUADRATIC FUNCTION

1. By looking at the data generated for each function in Activity One, indicate the vertex of each function.

a) __________

b) __________

c) __________

d) __________

e) __________

f) __________

g) __________

2. By inspecting the graphs of the following functions record the vertex and axis of symmetry of each function.

		Vertex	Axis
a)	$f(x) = x^2$	__________	__________
b)	$f(x) = x^2 + 4x + 3$	__________	__________
c)	$f(x) = 2x^2 - 4x + 4$	__________	__________
d)	$f(x) = 3x^2 + 6x + 6$	__________	__________
e)	$f(x) = -x^2 - 2x - 2$	__________	__________
f)	$f(x) = -2x^2 + 8x - 9$	__________	__________
g)	$f(x) = -3x^2 + 6x - 1$	__________	__________

Compare the verticies indicated in Question One and Question Two. Since the functions are exactly the same, hopefully you will have found that the verticies are also the same.

3. Based on your observations from Question Two, how can you determine the axis of symmetry of a quadratic function given the coordinates of the vertex?

4. If you are given two symmetric points on a graph of a quadratic function, how can you determine the axis of symmetry of the function. (HINT: If you were told that the points (8,10) and (2,10) lie on the graph of a quadratic function, how could you determine that the axis of symmetry is x = 5?)

5. Based on your responses to Questions Three and Four, answer the following questions:

   a) The graph of a quadratic function has a minimum point and contains the points (2,0) and (-2,0). What is the x-coordinate of the vertex, and what is the axis of symmetry?

   b) The graph of a quadratic function has a minimum point and contains the points (2,10) and (6,10). What is the x-coordinate of the vertex, and what is the axis of symmetry?

   c) The graph of a quadratic function contains points (3,8) and (5,8). What is the x-coordinate of the vertex, and what is the axis of symmetry?

d) The graph of a quadratic function has a maximum point and contains the points (-6,3) and (-1,3). What is the x-coordinate of the maximum point, and what is the axis of symmetry?

e) If (a,b) and (c,b) are on the graph of a quadratic function, what is the x-coordinate of the maximum or minimum point, and what is the equation of the axis of symmetry?

GRAPHS OF QUADRATIC FUNCTIONS

The major purpose of the activities included in this section is for students to establish further familiarity with the shape of the graph of a quadratic function and to understand the effect the parameters a, p, and q have on the graph of the function represented by the equation $f(x) = a(x - p)^2 + q$. The activities progressively examine the effect of the parameters by first observing the graphs of functions written in the form $f(x) = ax^2$, followed by an examination of graphs of functions written in the form $f(x) = ax^2 + q$, and then of graphs of functions written in the form $f(x) = a(x - p)^2 + q$. On completion of these activities students should have reached the following conclusions about the graph of a quadratic function represented by the equation $f(x) = a(x - p)^2 + q$:

a) the function will have a maximum point if $a < 0$, and a minimum point if $a > 0$;
b) if $a > 0$ then the graph is concave upward and if $a < 0$, then the graph is concave downward;
c) changing the value of q raises or lowers the graph along the y axis without changing its shape;
d) changing the value of p moves the graph horizontally along the x axis without affecting its shape; and,
e) the vertex of the function is (p,q).

In order to achieve these objectives, all three activities should be completed.

Suggested Approach

For the activities to be most beneficial, it is recommended that the students complete each activity prior to any class discussion of the graphs and their equations. In addition, each activity should be followed by a discussion of the graphs and equations under investigation to consolidate and verify the conclusions the students have reached.

These activities may be completed by each student individually or in small groups depending upon the equipment available.

Suggested Software

Lund, C. & Andersen, E.D. **Computer graphing experiments** (Vol. 1 and 3). Don Mills, Ontario: Addison-Wesley, 1982.

NOTE: Using this software package, Activities Three and Four can be completed by choosing the Parabola Option (Option 2) from the main menu page and then the appropriate sub-option from the second menu page. The functions may then be graphed by entering the appropriate parameters.

Activity Five is completed by choosing the Special Functions Option (Option 5) from the main menu page and then the option which allows you to enter your own special function (Option 5) from the second menu page. The functions in this activity may be graphed by entering the equation using the BASIC commands for multiplication, exponentiation, etc. The maximum and minimum values for both the domain and the range will be requested: -10 for the minimum value and 10 for the maximum value seems to work well in both instances.

ACTIVITY THREE

GRAPHING FUNCTIONS OF THE FORM $F(X) = AX^2$

1. Use the same coordinate axis to sketch the graphs of each of the following quadratic functions:

   $f(x) = x^2$,

   $f(x) = 2x^2$,

   $f(x) = 3x^2$,

   $f(x) = 1/2\ x^2$, and

   $f(x) = 1/3\ x^2$.

   a) How are the graphs similar?

   b) How do the graphs differ?

2. Use the same coordinate axis to sketch the graphs of each of the following quadratic functions:

   $f(x) = -x^2$,

   $f(x) = -2x^2$,

   $f(x) = -3x^2$,

   $f(x) = -1/2\ x^2$, and

   $f(x) = -1/3\ x^2$.

   a) How are the graphs similar?

   b) How do the graphs differ?

3. a) What effect does increasing the $|a|$ have on the graph of $f(x) = ax^2$?

   b) What is the effect of decreasing the $|a|$?

4. Use the same coordinate axis to sketch the graphs of each pair of functions.

   a) $f(x) = x^2$ and $f(x) = -x^2$

   b) $f(x) = 3x^2$ and $f(x) = -3x^2$

   c) $f(x) = 1/4\ x^2$ and $f(x) = -1/4\ x^2$

   Answer the following questions for each pair of functions:

   i) What do the two functions have in common?

   ii) In what way do the two functions differ?

   iii) What is the relationship between the two functions?

5. The quadratic functions you have sketched have been represented by equations of the form $f(x) = ax^2$ where a is an element of R and $a \neq 0$.

a) What is the vertex of these functions?

b) What is the axis of symmetry of these functions?

c) What is the domain of these functions?

d) The following questions refer to the function $f(x) = ax^2$ where $a > 0$.

i) In which direction will this graph open?

ii) What is the range of the function?

iii) Is the vertex a maximum or minimum?

e) The following questions refer to the function $f(x) = ax^2$ where $a < 0$.

i) In which direction will this graph open?

ii) What is the range of the function?

iii) Is the vertex a maximum or minimum?

ACTIVITY FOUR

GRAPHING FUNCTIONS OF THE FORM $F(X) = AX^2 + Q$

1. Use the same coordinate axis to sketch the graph of each of the following quadratic functions:

   $f(x) = x^2$,

   $f(x) = x^2 + 2$, and

   $f(x) = x^2 - 2$.

   a) What is the vertex of each function?

   b) What is the axis of symmetry of each function?

   c) How are the graphs similar?

   d) How do the graphs differ?

2. Use the same coordinate axis to sketch the graph of each of the following quadratic functions:

   $f(x) = -x^2$,

   $f(x) = -x^2 + 2$, and

   $f(x) = -x^2 - 2$.

   a) What is the vertex of each function?

   b) What is the axis of symmetry of each function?

c) How are the graphs similar?

d) How do the graphs differ?

3. Use the same coordinate axis to sketch the graph of each of the following quadratic functions. Record the vertex and axis of symmetry of each function.

	Vertex	Axis
a) $f(x) = x^2 + 5$		
b) $f(x) = -x^2 + 5$		
c) $f(x) = -2x^2 + 5$		
d) $f(x) = 1/4\ x^2 + 5$		

4. Use the same coordinate axis to sketch the graph of each of the following quadratic functions. Record the vertex and axis of symmetry of each function.

	Vertex	Axis
a) $f(x) = 3x^2 - 4$		
b) $f(x) = -1/2\ x^2 - 4$		
c) $f(x) = x^2 - 4$		
d) $f(x) = -2x^2 - 4$		

5. Without graphing the functions, answer the following questions based upon your observations from the previous questions. The functions you have sketched were all expressed by equations of the form $f(x) = ax^2 + q$ where a and q are elements of R and $a \neq 0$.

   a) Sketch the position of the graph of a quadratic function with the following parameters:

      i) $a > 0$ and $q > 0$　　ii) $a > 0$ and $q < 0$

      iii) $a < 0$ and $q > 0$　　iv) $a < 0$ and $q < 0$

   b) What effect does increasing the value of q have on the graph of the function $f(x) = ax^2 + q$?

   c) What effect does decreasing the value of q have on the graph of the function $f(x) = ax^2 + q$?

6. a) What is the difference between the graph of $f(x) = 3x^2 - 5$ and the graph of $f(x) = 3x^2 + 4$?

   b) What do these two graphs have in common?

   After answering the question, check your answer by sketching the graphs.

ACTIVITY FIVE

GRAPHING FUNCTIONS OF THE FORM $F(X) = A(X - P)^2 + Q$

1. Sketch each of the following sets of graphs on the same coordinate axis.

		Vertex	Axis
a)	$f(x) = x^2$		
	$f(x) = (x - 2)^2$		
	$f(x) = (x + 3)^2$		
b)	$f(x) = 2x^2 - 1$		
	$f(x) = 2(x - 4)^2 - 1$		
	$f(x) = 2(x + 3)^2 - 1$		
c)	$f(x) = -3x^2 + 2$		
	$f(x) = -3(x + 1)^2 + 2$		
	$f(x) = -3(x - 2)^2 + 2$		
d)	$f(x) = (x + 1)^2$		
	$f(x) = 4(x + 1)^2$		
	$f(x) = -2(x + 1)^2$		
e)	$f(x) = 1/2\ (x + 3)^2 - 1$		
	$f(x) = 1/2\ (x + 3)^2 - 4$		
	$f(x) = 1/2\ (x - 2)^2 + 2$		
f)	$f(x) = (x + 3)^2 - 4$		
	$f(x) = 1/2\ (x - 1)^2$		
	$f(x) = -2(x + 4)^2 - 3$		

2. The functions in the previous question were all expressed by equations written in the form $f(x) = a(x - p)^2 + q$.

   a) What are the values of p and q for each function?

   b) What is the relationship between the x and y coordinates of the vertex and the variables p and q in the equation of the function?

   c) What is the relationship between the value of 'a' in the equation and the graph of the function?

3. In the equation $f(x) = a(x - p)^2 + 1$, what values of a, p and q will produce graphs with the following characteristics:

   a) vertex at the origin and opening upward?

   b) vertex anywhere on the x axis, other than the origin, and opening downward?

   c) opening upward and crossing the x axis in two points?

   d) opening downward and crossing the x axis in two points?

COMPLETING THE SQUARE

The actual procedure of completing the square does not lend itself to computer application other than through the use of a tutorial or CAI software package designed to improve the students' ability to apply and complete the algorithm. For this reason a teacher-centered approach to the topic is suggested.

The following activity is designed as an additional or follow-up activity to the actual procedure of completing the square. Specifically, this activity was designed to reinforce the students' understanding of how the variables a, b, and c, in the equation $f(x) = ax^2 + bx + c$, determine the coordinates of the vertex, the equation of the axis of symmetry, and the maximum or minimum value of the graph.

Suggested Approach

Activity Six should be assigned only after students have investigated completing the square of the general case ($f(x) = ax^2 + bx + c$) and have established how the coordinates of the vertex, the axis of symmetry, and the maximum or minimum value can be determined from the variables a, b, and c. Students should be familiar with the following formulas:

a) equation of the axis of symmetry: $x = \frac{-B}{2A}$,

b) maximum or minimum value: $\frac{4AC-B^2}{4A}$

c) the coordinates of the vertex: $\frac{-B}{2A}, \frac{4AC-B^2}{4A}$

Students should be encouraged to test their programs by completing a few examples by hand and then using the same functions as data for their programs. In addition, students should also be reminded to check the data that their program accepts and to be particularly aware of the restriction that exists on the value of 'a'.

ACTIVITY SIX

COMPLETING THE SQUARE

1. Write a program that given the values of a, b, and c, in the equation $f(x) = ax^2 + bx + c$, determines and prints the equation of the axis of symmetry, the coordinates of the vertex, the maximum or minimum value, and specifies if the value is a maximum or a minimum. (HINT: Begin your program with the statement 10 INPUT A,B,C and be sure to check the variables on which restrictions exist.)

2. Use your program to determine the axis of symmetry, the vertex, and the maximum or minimum value of the following functions.

Function	Axis of Symmetry	Vertex	Max. or Min. (Please specify)
a) $f(x) = x^2 - 4x + 8$			
b) $f(x) = x^2 - 2x + 3$			
c) $f(x) = 2x^2 - 4x + 3$			
d) $f(x) = -3x^2 + 6x$			
e) $f(x) = 4x^2 - 4$			
f) $f(x) = 1 - x - x^2$			
g) $f(x) = 13x + 1$			
h) $f(x) = 1 - x^2$			
i) $f(x) = 3x^2 + 3x$			
j) $f(x) = -12x + 2x^2 -1$			

ROOTS OF A QUADRATIC FUNCTION

The activities in this section are designed to provide students with a variety of experiences in determining the roots of a quadratic equation. Specifically, the activities involve determining the roots of an equation by:

a) successively narrowing and searching the interval which contains the root;
b) conducting a binary search of the interval; and,
c) applying the quadratic formula.

By experiencing a variety of approaches, students should acquire a better understanding of the concept of roots of an equation.

Suggested Approach

The methods of determining roots studied in Activities Seven and Eight are based upon the location principle which states that if a graph is continuous in the interval a < = x < = b, and the y values at x = a and x = b have opposite signs, then there exists at least one root between x = a and x = b. To ensure that students understand these first two approaches to determining roots, it is suggested that these activities be preceded by a discussion of the location principle after which students would be free to work on the activities at their own pace. If time does not permit, Activities Seven and Eight could be assigned to various groups or completed as a class assignment. An attempt should be made to include at least one of the activities.

Activity Nine centers on writing a BASIC program to calculate the roots of a quadratic function through the use of the quadratic formula. Students should be encouraged to use the functions and the roots that were determined in the previous two exercises to test their programs prior to using the program to determine the roots of the functions included in this specific exercise. In the following activity, Activity Ten, students are asked to extend their programs to include in the printout the sum and the product of the roots of the quadratic function under investigation. By examining the resulting data, students should conclude that the sum of the roots of a quadratic function is given by -B/A and that the product of the roots is given by C/A. Students are then asked to apply their conclusions in Question Four of the exercise.

ACTIVITY SEVEN

SEARCHING INTERVALS TO DETERMINE ROOTS

	No. of roots in the domain -10 to 10	For Determining Non-Integral Roots		Roots
		Integers the roots lie between	Next Interval	
1. $f(x) = x^2 - 4x - 5$				
2. $f(x) = 3x^2 - 10x - 8$				
3. $f(x) = 4x^2 - 0.25$				
4. $f(x) = x^2 + 0.7x - 0.1$				
5. $f(x) = -x^2 + 16$				
6. $f(x) = 2x^2 + 5x + 1$				
7. $f(x) = 2x^2 - 7x + 3$				
8. $f(x) = x^2 - 3.5x - 2$				
9. $f(x) = -x^2 + 2x + 3$				
10. $f(x) = -x^2 + 2x + 3$				

ACTIVITY EIGHT

A BINARY SEARCH TO DETERMINE THE ROOTS OF A QUADRATIC FUNCTION

In the previous activity, non-integral roots were determined by successively narrowing and examining the interval in which the root occurred. Another and sometimes faster approach is that of a binary search.

A binary search begins in a similar manner to the previous approach; however, after locating the root of the function between two integers, the interval is searched by dividing the interval in half and determining which half-interval crosses the x-axis. That is, the rule for selecting the half-interval to undergo further searching is to choose the interval that is negative at one end and positive at the other end. This will ensure that the graph crosses the x-axis in this interval. Once the appropriate half-interval is chosen, its midpoint is determined and the process is repeated to determine which new half-interval crosses the x-axis.

1. Knowing the endpoints of an interval, how can you determine if they fall on opposite sides of the x-axis, i.e., how can you determine if one endpoint is greater than zero and one endpoint is less than zero? (HINT: What facts do you know about combining negative and positive numbers that might be of help?)

2. Write a program that will use this procedure to determine the roots of a function in the domain $-10 <= x <= 10$.

3. Use the following functions to test your program:

   a) $f(x) = 4x^2 + 4x - 3$

   b) $f(x) = 2x^2 + 5x + 2$

   c) $f(x) = x^2 + 5x + 2$

   d) $f(x) = x^2 + x - 11$

   e) $f(x) = 6x^2 - 6x - 1$

ACTIVITY NINE

THE QUADRATIC FORMULA

1. Write a BASIC program that will use the quadratic formula to determine and print out the roots of any quadratic function. Begin your program with the statement 10 INPUT A,B,C and be sure to check for any restrictions that exist on the variables.

2. Use the functions and the roots determined in Activity Seven to determine if your program is working correctly.

3. Use your program to determine the roots of the following functions. Record the results.

a) $f(x) = 2x^2 - 8x + 6$ ____________________

b) $f(x) = x^2 + 5x - 84$ ____________________

c) $f(x) = x^2 + 6x + 4$ ____________________

d) $f(x) = 2x^2 - 4x + 3$ ____________________

e) $f(x) = 0x^2 + 0x + 5$ ____________________

f) $f(x) = 4x^2 - 9x + 5$ ____________________

g) $f(x) = 5x^2 - 17$ ____________________

h) $f(x) = -2x^2 + 4x + 6$ ____________________

i) $f(x) = x^2 - 2x - 1$ ____________________

j) $f(x) = 0x^2 + 2x + 6$ ____________________

k) $f(x) = x^2 + x + 1$ ____________________

l) $f(x) = 0x^2 + 0x + 0$ ____________________

ACTIVITY TEN

THE SUM AND THE PRODUCT OF THE ROOTS

1. Revise your program from Activity Nine so that it prints the values of A,B, and C in the functions, the two roots, the sum of the roots, and the product of the roots.

2. Use your program to determine the roots, the sum of the roots, and the product of the roots of the following functions. Record the results.

		A	B	C	Roots	Sum	Prod.
a)	$f(x) = x^2 + 4x - 2$						
b)	$f(x) = -x^2 + 3x - 2$						
c)	$f(x) = 2x^2 + 10x + 8$						
d)	$f(x) = 2x^2 - 2x - 12$						
e)	$f(x) = 3x^2 - 9x + 6$						
f)	$f(x) = -3x^2 - 15x - 12$						
g)	$f(x) = 3x^2 - 3x - 18$						

3. What relationship exists between the values of A,B, and C and the sum and the product of the roots of the functions?

4. Use your conclusions from Question Three to answer the following questions:

   a) If 2 is one root of the function $f(x) = x^2 - 5x + c$, find the other root and the value of c.

   b) If -1/2 is one root of the function $f(x) = 4x^2 - 3x - c$, find the other root and the value of c.

   c) If 3/2 is one root of the function $f(x) = 4x^2 + bx - 3$, find the other root and the value of b.

   d) Given that the roots of the function $f(x) = 4x^2 - 16x + c$ are equal, find the value of c.

THE NATURE OF THE ROOTS

The discriminant of a quadratic function provides important information about the roots of the function which, in turn, provide important information about the shape of the graph of the function. The following activity is designed to assist students in determining the relationships that exist between the value of the discriminant, the type of roots the function has, and the shape of the graph of the function. Specifically, on completion of the activity, students should understand the following:

a) if the value of the discriminant is greater than zero, then the function has two [illegible]al roots, and the graph of the function will cut the x axis in two distinct points;
b) if the value of the discriminant is equal to zero, then the quadratic function has equal roots, and the graph of the function touches the x axis; and,
c) if the value of the discriminant is less than zero, then the quadratic function has no real roots, and the graph does not intersect the x axis.

The activity is optional for this unit.

Suggested Approach

In Activity Eleven students are asked to revise their quadratic formula program from Activity Nine so that it will include the value of the discriminant, in addition to the roots of the function, in the printout. Students are then asked to investigate and draw conclusions about how the value of the discriminant can be used to determine the number of roots the quadratic function has. Following this investigation, students are asked to use the computer to graph the functions to determine how the value of the discriminant and the number of roots is reflected in the shape of the function's graph. The activity concludes by asking students to graph a series of functions and to determine the value of the discriminant of each function based on the shape of the graph and the conclusions they have reached earlier.

This particular activity could be worked on by students individually or in pairs. If the pair arrangement is chosen it might be effective to have one student use the computer to run the revised quadratic formula program, to determine the number of roots the function has and the value of the discriminant, and the other student could use the computer to graph the same function. By working in pairs students would then have the added benefit of seeing the whole picture at once, and the length of time to complete the activity would be reduced. It is strongly recommended that a class discussion be held after students have had sufficient time to complete the activity to ensure that students have reached the proper conclusions.

Suggested Software

Lund, C. & Andersen, E.D. **Computer graphing experiments** (Vol. 1 and 3). Don Mills, Ontario: Addison-Wesley, 1982.

ACTIVITY ELEVEN

THE NATURE OF THE ROOTS

1. The expression $B^2 - 4AC$ in the quadratic formula is called the discriminant of the function $f(x) = ax^2 + bx + c$. Revise your quadratic formula program from Activity Nine so that it will print out the value of the discriminant in addition to the roots of the function.

2. Use your program to determine the roots and the value of the discriminant of each of the following functions.

a) $f(x) = 4x^2 - 20x + 25$ ____________ ____________

b) $f(x) = 2x^2 - x - 3$ ____________ ____________

c) $f(x) = 4x^2 - 3x + 5$ ____________ ____________

d) $f(x) = 5x^2 - 5x + 1$ ____________ ____________

e) $f(x) = 3x^2 - 7x + 3$ ____________ ____________

f) $f(x) = x^2 + 2$ ____________ ____________

g) $f(x) = x^2 + 3x - 5$ ____________ ____________

h) $f(x) = 25x^2 + 10x + 1$ ____________ ____________

i) $f(x) = 3x^2 - x + 1$ ____________ ____________

j) $f(x) = 9x^2 - 6x + 1$ ____________ ____________

3. What relationship do you observe between the number of real roots the function has, and the value of the discriminant? (HINT: How may real roots are there when the discriminant is less than zero? equal to zero? greater than zero?)

4. Use the computer to graph the functions that are listed in Question Two. What relationship do you see between the shape of the graph, the value of the discriminant, and the number of roots the function has?

5. Use the computer to graph the following functions. By observing the shape of the graph, specify how many roots the function will have and whether the discriminant will be greater than, less than or equal to zero.

a) $f(x) = x^2 - 4x + 4$ __________ __________

b) $f(x) = 4x^2 - 12x + 9$ __________ __________

c) $f(x) = x^2 + x + 1$ __________ __________

d) $f(x) = -x^2 - 4$ __________ __________

e) $f(x) = 4 - x^2$ __________ __________

f) $f(x) = 9x^2 + 12x - 4$ __________ __________

APPENDIX B

Pre-test and Post-test Results

QUADRATIC FUNCTIONS PRE-TEST

RESPONSE ANALYSIS (N=15)

1. Evaluate the quadratic function $f(x)=x^2+3x-2$ at $x=5$.

Response	Frequency
38	14
28	1

Method	Frequency
Substitution	15

2. Without calculating the values of the function $f(x)=3x^2+9x+2$, identify any error(s) in this table of values. Explain your answer.

x	f(x)
-5	32
-4	14.3
-3	2
-2	-4
-1	-4
0	2
1	14
2	36

Response	Frequency	Explanation	Frequency
14.3	12	Must be an integer	10
		Substitution	1
		No explanation	1
36	7	Substitution	5
		Symmetric about 0	2
No response	1		

3. Does the point (2,4) lie on the curve of the function $f(x)=2x^2-3x+2$? Why or why not?

Response	Frequency	Method	Frequency
Yes	14	Substitution	13
No response	1	Quadratic formula	1

4. What values of k will make the following trinomials perfect squares?

a) $x^2+kx+36$

Response	Frequency
12	13
±12	1
$(x+9)(x+4)$	1

b) $9x^2+12x+k$

Response	Frequency
4	12
$(3x+6)^2$	1
3	1
$(3x)(3x)$	1

5. Is the set {1/3, -1/2} a solution of the equation $6x^2-5x+1=0$? Why or why not?

Response	Frequency	Method	Frequency
Yes	2	Substitute (only 1/3)	2
No	12	Substitution	8
		Factoring	3
		Quadratic formula	1
No response	1		

6. Solve the following equations. Explain your procedure.

a) $3x^2-75=0$

	Method	
Response	Factored Quadratic	Solved the Equation
5	0	1
±5	10	2
5 and 0	1	0
Incorrect	0	1

b) $6x^2+2x=0$

	Method	
Response	Factored Quadratic	Solved the Equation
-1/3 and 0	8	1
Only 0	2	0
Incorrect	3	0

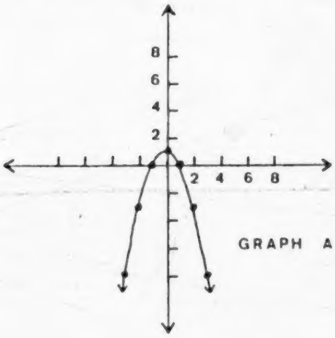
Note: One student did not attempt this question.

c) $x^2-4x-12=0$

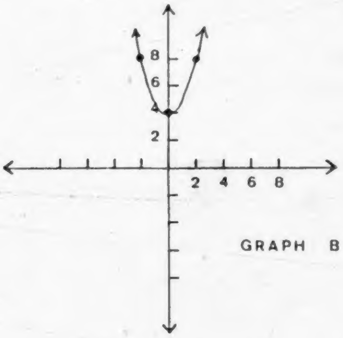
	Method		
Response	Factored Quadratic	Solved the Equation	Quadratic Formula
6 and -2	12	0	1
±6, ±2	1	0	0
-6 and +2	1	0	0

7. a) Match each of the equations with the appropriate graph.

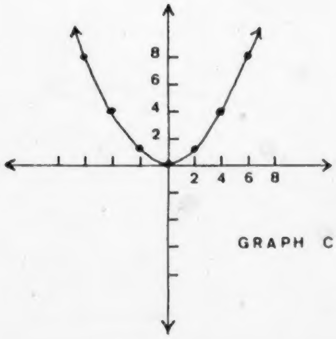
Equation	Graph	Correct Responses	No Response
$f(x)=x^2+4$	B	15	0
$f(x)=.25x^2$	C	15	0
$f(x)=-4x^2$	F	15	0
$f(x)=-x^2+1$	A	15	0
$f(x)=(x+1)^2-4$	D	14	1
$f(x)=(x-4)^2+1$	E	14	1



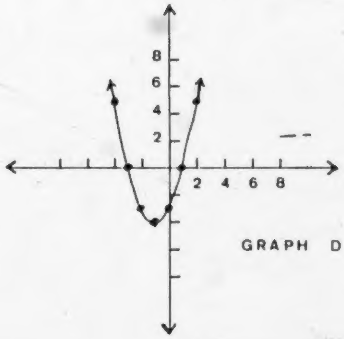
GRAPH A



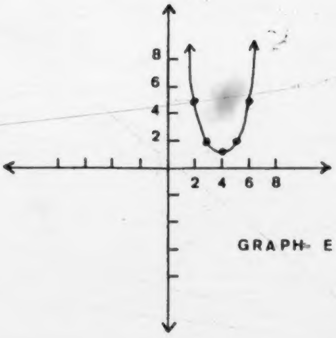
GRAPH B



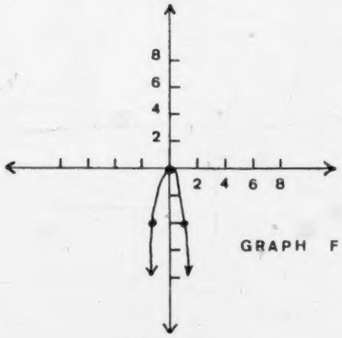
GRAPH C



GRAPH D



GRAPH E



GRAPH F

b) How did you determine which equation matched each graph?

Method	Frequency
Substitution (x=0)	7
Table of values	2
Max./min. points	1
No method stated	5

QUADRATIC FUNCTIONS POST-TEST

RESPONSE ANALYSIS (N=15)

1. a) Draw a sketch of the graph of the function $f(x)=ax^2+c$ where $a>0$ and $c>0$.

Response	Frequency
Correct	14
Incorrect	1

b) At what point does the graph cross the x axis?

Response	Frequency
Never	15

2. a) How would it be possible for a function of the form $f(x)=ax^2+c$ to have no x-intercepts even though the value of c is less than 0?

Response	Frequency
a is negative	11
a is not negative	1
c is less than 0	1
$f(x)=x+-1$	1
No response	1

b) Draw a sketch of the graph.

Response	Frequency
Correct	13
No response	2

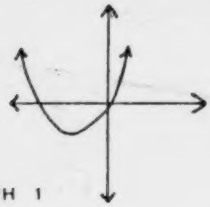
3. Would it be possible for the graph of a quadratic function to have (0,0) as a minimum point and also contain the points (1,3) and (-1,1). Explain your answer.

Response	Frequency	Explanation	Frequency
No	11	Not symmetric,	9
		(-1,1) lower than minimum	2
Yes	2	Range ≥0, domain R	1
		The points are above vertex	1
No response	3		

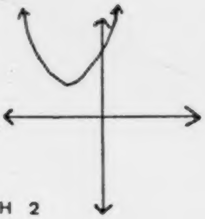
4. Would it be possible for the graph of a quadratic function to have (-8,-10) as a maximum point and also contain the points (0,-11) and (16,-11). Explain your answer.

Response	Frequency	Explanation	Frequency
Yes	9	Points below vertex	3
		Symmetric	6
No	3	maximum point below (0,-11)	2
		Not symmetric	1
No response	3		

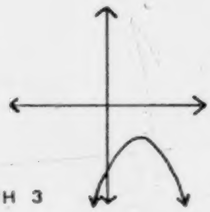
5. Each of the following graphs can be represented by an equation of the form $f(x)=a(x-h)^2+k$. Match each graph on the left with the appropriate values of a, k, and h.



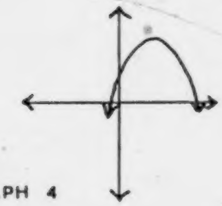
GRAPH 1



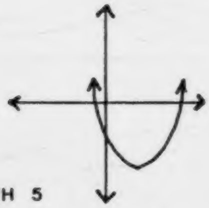
GRAPH 2



GRAPH 3



GRAPH 4



GRAPH 5

(A) $a>0$, $k>0$, $h>0$

(B) $a>0$, $k>0$, $h<0$

(C) $a>0$, $k<0$, $h>0$

(D) $a>0$, $k<0$, $h<0$

(E) $a<0$, $k<0$, $h<0$

(F) $a<0$, $k>0$, $h>0$

(G) $a<0$, $k>0$, $h<0$

(H) $a<0$, $k<0$, $h>0$

The distributions of responses for each of the graphs in Question Five are presented in the following table.

	Values							
Graph	A	B	C	D	E	F	G	H
1	0	0	5	10	0	0	0	0
2	5	9	1	0	0	0	0	0
3	0	0	0	0	4	0	1	10
4	0	0	0	0	0	11	4	0
5	0	2	10	3	0	0	0	0

6. Determine the value of A for the function $f(x)=Ax^2+8x-3$, if the graph of the function is symmetric about the line $x=2$.

Response	Frequency
a=-2	2
-b/2a=2	5
-3.25	3
-b/2a=?	2
y=4a+16-3	1
-b/2a=-2	1
No response	1

7. The following data were determined by evaluating the function $f(x)=x^2-x-12$ in the domain $-5 \le x \le 5$. Without calculating, can you determine what is wrong with the data?

x	f(x)
-5	18
-4	8
-3	0
-2	- 6
-1	-10
0	0
1	-12
2	-10
3	- 6
4	0
5	8

Response	Frequency
Cannot have 3 roots	10
3 roots, not symmetric	5

8. The following data were determined by evaluating the function $f(x)=x^2-4x+2$ in the domain $-5 \le x \le 6$. Without calculating, can you determine what is wrong with the data?

x	f(x)
-5	47
-4	34
-3	21
-2	14
-1	8
0	2
1	- 1
2	-22
3	- 1
4	2
5	7
6	14

Response	Frequency
Not symmetric	11
Y's don't correspond	2
Sub. -1 to check	1
Lines, not parabola	1

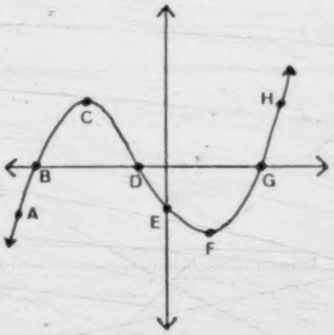
9. The data below were determined by evaluating the function $f(x)=4x^3-3x-2$ in the domain $-4 \le x \le 4$. Does this function have a root in the domain $-1 \le x \le 0$? Explain your answer.

x	f(x)
-4	50
-3	25
-2	8
-1	- 1
0	- 2
1	5
2	20
3	43
4	76

Response	Freq.	Explanation	Freq.
No	9	Does not cross the x axis	9
Yes	3	Changes from + to -	1
		Crosses the x axis	1
		No 0 in f(x)	1
No response	3		

10. Use your knowledge of quadratic functions to indicate which point(s) might be roots of this cubic function. Explain your answer.

Response	Frequency
A	0
B	8
C	0
D	6
E	1
F	0
G	8
H	0



Explanation	Frequency
Where graph cuts x axis	6
Where y=0	1
Where x=0	1
Must be symmetric	1
Must have 3 roots	~~1~~
Must have 4 roots	1
Not the third root	1
No response	3

11. Determine the value of B that allows the graph of $f(x)=-3x^2+Bx+16$ to cross the x axis at both $x=-2$ and $x=1$.

Response	Frequency
B=-3	10
Incomplete	5

Method	Frequency
Use of the formula $x= - b/2a$	7
Substitution	5
Quadratic formula	1
Expanded the factors $(x+2)(x-1)$	1
$x = \sqrt{B}/3\ x + 2$	1

12. Determine, without solving, whether the set {-9, -1} is the solution set of the equation $x^2-8x-9=0$.

Response	Frequency	Method	Frequency
No	14	Substitution	11
		Quadratic formula	1
		9 must be positive	1
		No explanation	1
No response	1		

13. A function f, where $f(x)=ax^2+bx+c$, $a\neq 0$, is described in each question. State whether the graph of the function has a maximum or a minimum value, and tell how many points on the x axis are also on the graph of the function.

a. $a<0$, and $b^2-4ac<0$

Response	Frequency	Response	Frequency
Maximum	14	0	13
Minimum	0	1	0
No response	1	2	1
		No response	1

b. $a>0$, and $b^2-4ac>0$

Response	Frequency	Response	Frequency
Maximum	0	0	1
Minimum	14	1	1
No response	1	2	10
		No response	3

c. $a>0$, and the function has equal roots

Response	Frequency	Response	Frequency
Maximum	0	0	0
Minimum	14	1	8
No response	1	2	4
		No response	3

d. $b^2-4ac>0$, $b=0$, and $c>0$

Response	Frequency	Response	Frequency
Maximum	9	0	1
Minimum	1	1	1
No response	5	2	9
		No response	4

APPENDIX C

Student Questionnaire

Student Questionnaire

As you are probably aware, we have been using the computer in your math class as part of a research project I am conducting while studying at the University. The research project was designed to investigate the use of computers in mathematics and to concentrate specifically on the following areas:

1. to determine how computers can be used in mathematics;
2. to determine if the use of computers helps students learn mathematics any better than the regular approaches to mathematics; and,
3. to determine how the students and teacher involved in the project feel about using the computer in mathematics.

The purpose of this questionnaire is to gain feedback from you in regards to the third point listed above.

Please answer the following questions as truthfully and candidly as possible. I would appreciate any feedback you can give me above and beyond the specific questions included in this booklet.

If you do not feel comfortable answering the questions, please submit the booklet as is.

This is not a test of any sort, and the information will not be used in any way to determine your grade in this course.

The purpose of the following questions is for you to give me your general opinions about computers and programming. There are no right or wrong answers. Please circle the response that best describes how you feel about the statements. The responses are:

SD -- Strongly Disagree
D -- Disagree
N -- No opinion
A -- Agree
SA -- Strongly Agree

a) I enjoyed using the computer.	SD	D	N	A	SA
b) I feel confident about my ability to use computers.	SD	D	N	A	SA
c) The time I spent on the computer was worthwhile.	SD	D	N	A	SA
d) Writing computer programs helped me learn.	SD	D	N	A	SA
e) Using the computer helped me learn more about math.	SD	D	N	A	SA
f) I would like to use the computer more in math.	SD	D	N	A	SA
g) I prefer to work by myself on the computer.	SD	D	N	A	SA
h) I enjoyed writing computer programs more than I enjoyed using the prepared programs like the graphing programs.	SD	D	N	A	SA
i) There was adequate time to complete the assignments.	SD	D	N	A	SA
j) Having two people teach the unit was confusing.	SD	D	N	A	SA
k) The regular teacher should teach both the classroom and the microcomputer sessions.	SD	D	N	A	SA
l) Every student should have some minimal understanding of computers.	SD	D	N	A	SA
m) Every secondary school student should be able to write a simple computer program.	SD	D	N	A	SA

n) Computers can be useful instructional aids in many subject areas other than mathematics. SD D N A SA

o) The use of computers in education results in less personal treatment of students. SD D N A SA

p) Computers are gaining too much control over people's lives. SD D N A SA

2. Before using the computer in math, did you have any previous experience with computers?

( ) Yes ( ) No

If you answered yes, please explain where you have used a computer before and the amount of involvement you had.

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3. Did you have any difficulty operating the computer?

( ) Yes ( ) No

If yes, please explain the kind of difficulties you had.

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4. What did you like most about using the computer in math?

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5. What did you like least about using the computer in math?

6. Which assignment(s) did you like the most? Why?

7. Which assignments did you like the least? Why?

8. The following space is provided for you to make any additional comments about the project or the use of computers in mathematics.

Thank you very much for your help during the past month and a half through your involvement in the project. I have really enjoyed working with you.

Karen Miklos